The Computational Advantage of Depth: Learning High-Dimensional Hierarchical Functions with Gradient Descent

The Computational Advantage of Depth: Learning High-Dimensional Hierarchical Functions with Gradient Descent

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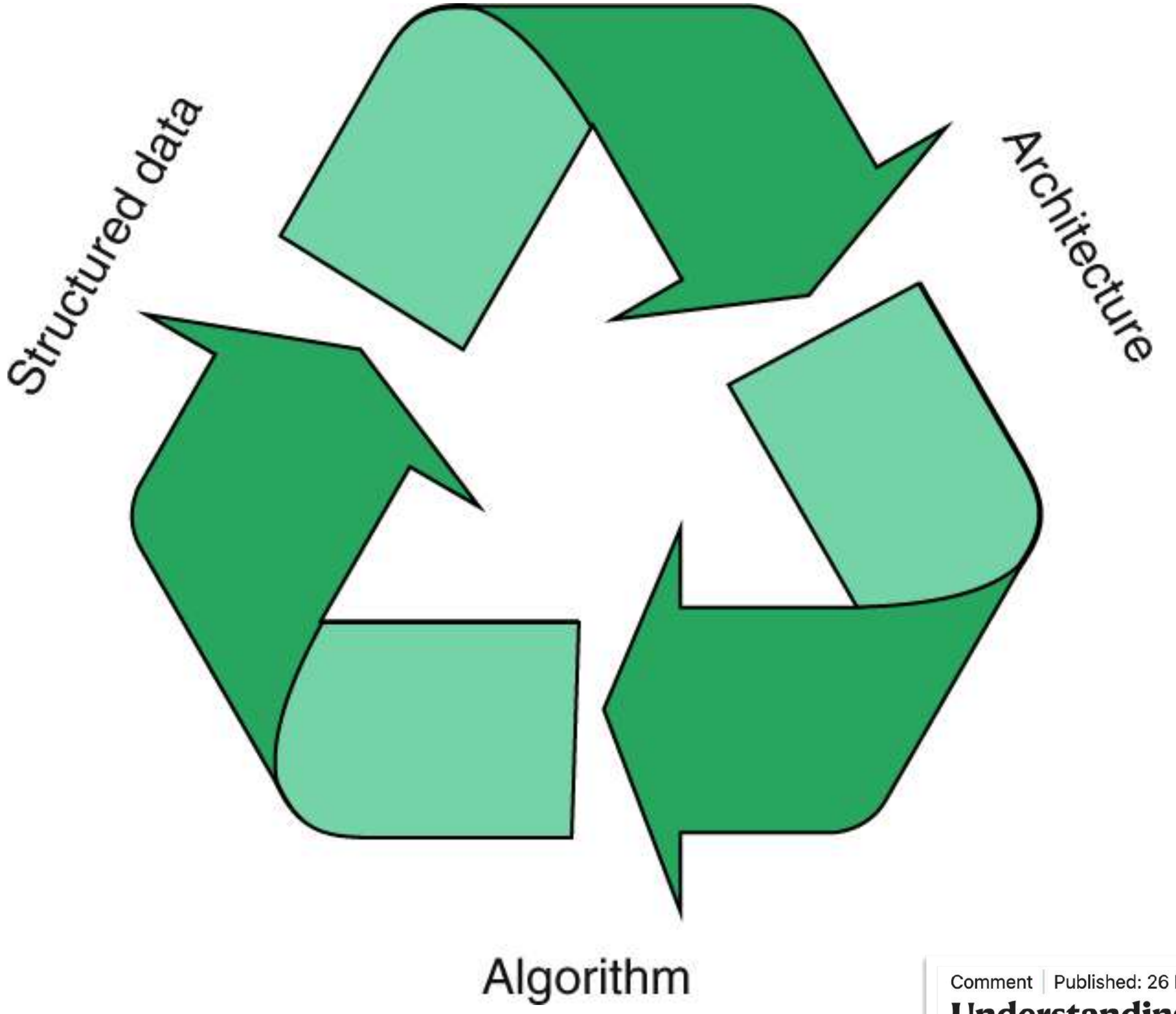
INFORMATION, LEARNING & PHYSICS LAB.

SPOC
Statistical Physics Of Computation

Understanding deep learning is also a job*



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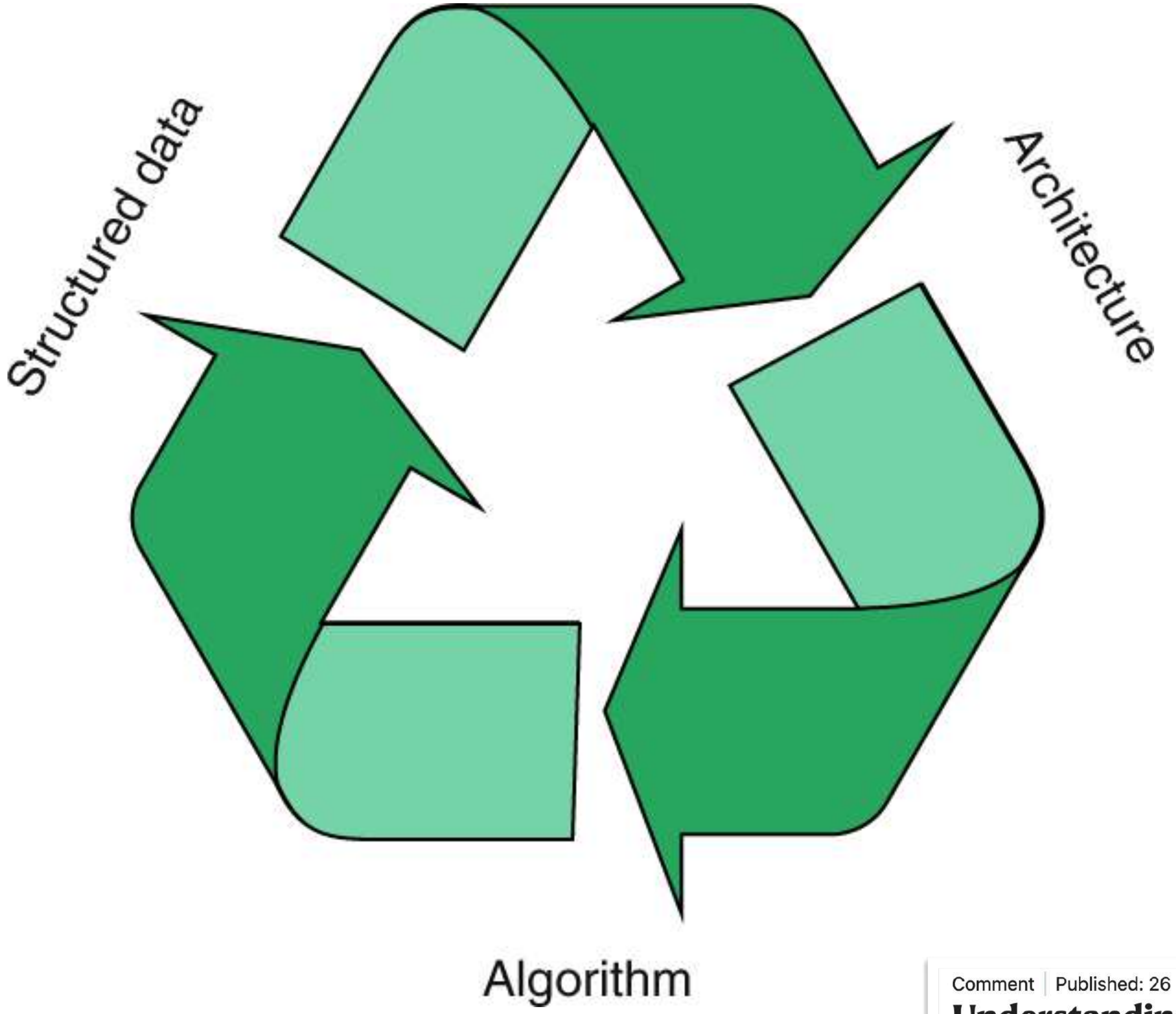

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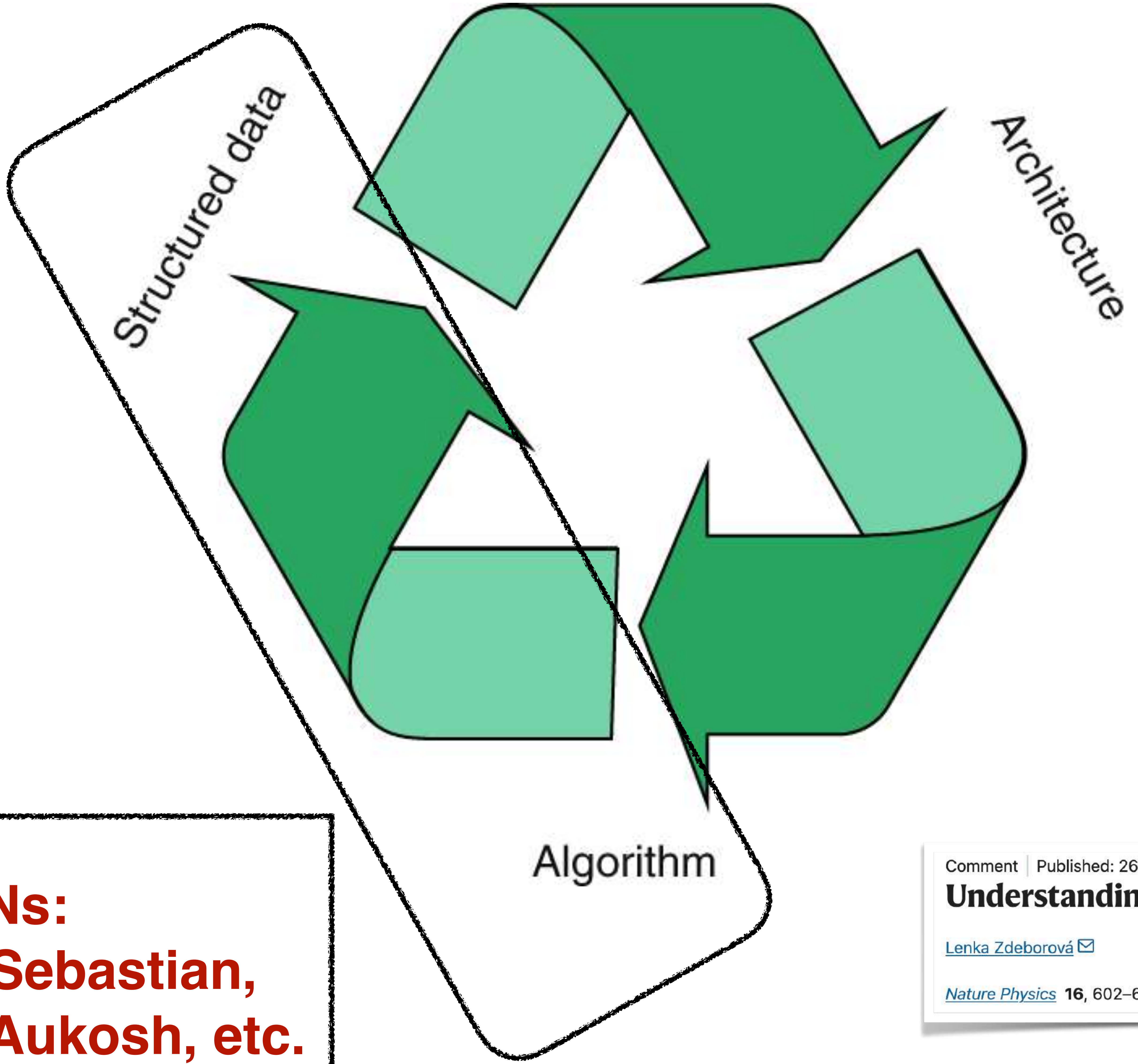

2-layer NNs:
Bruno’s lecture, Sebastian, Joan, Inbar, Alex, Aukosh, etc.


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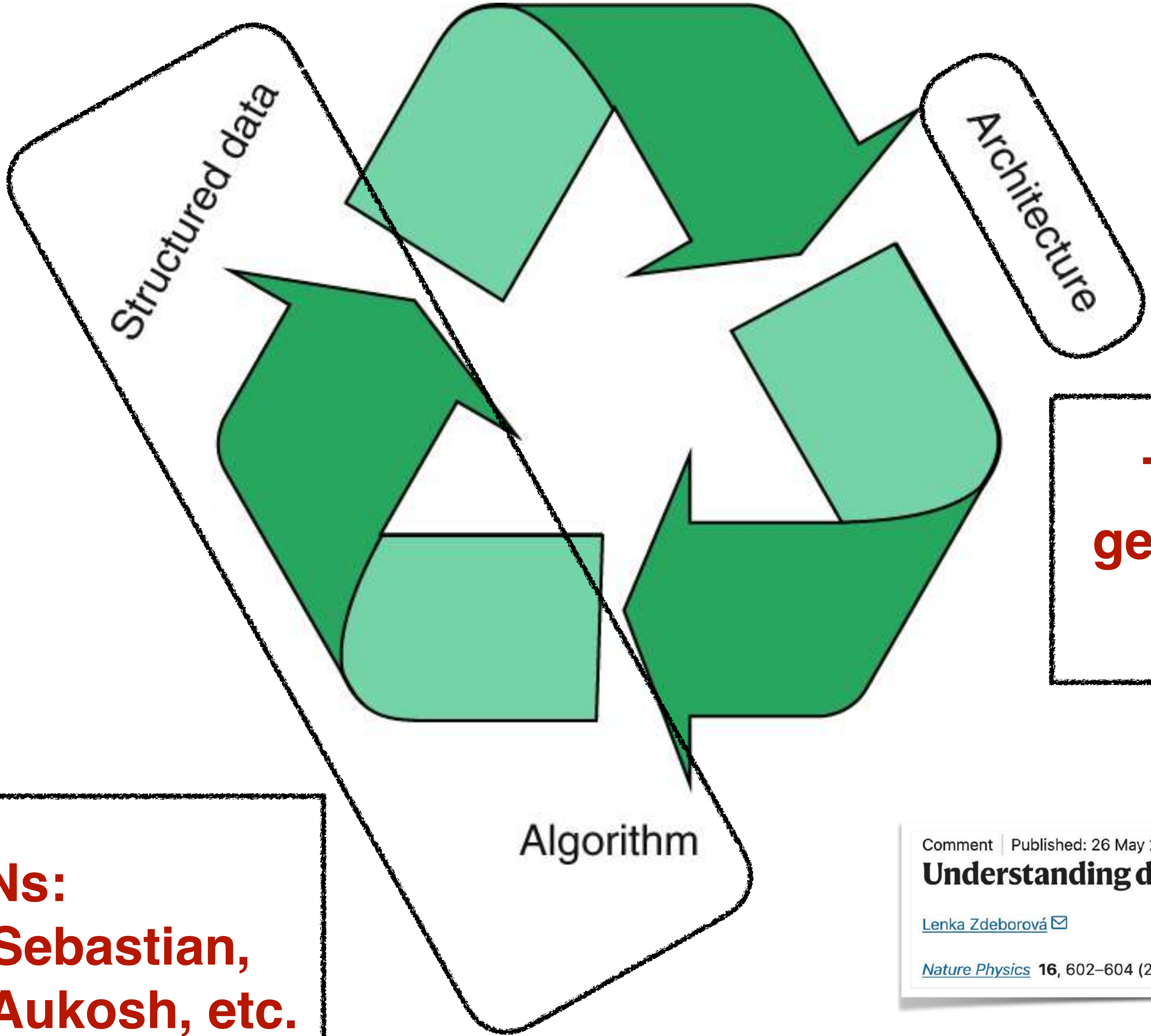

Transformers, CNNs, geometric deep learning, etc.

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It's called *deep* learning for a reason.

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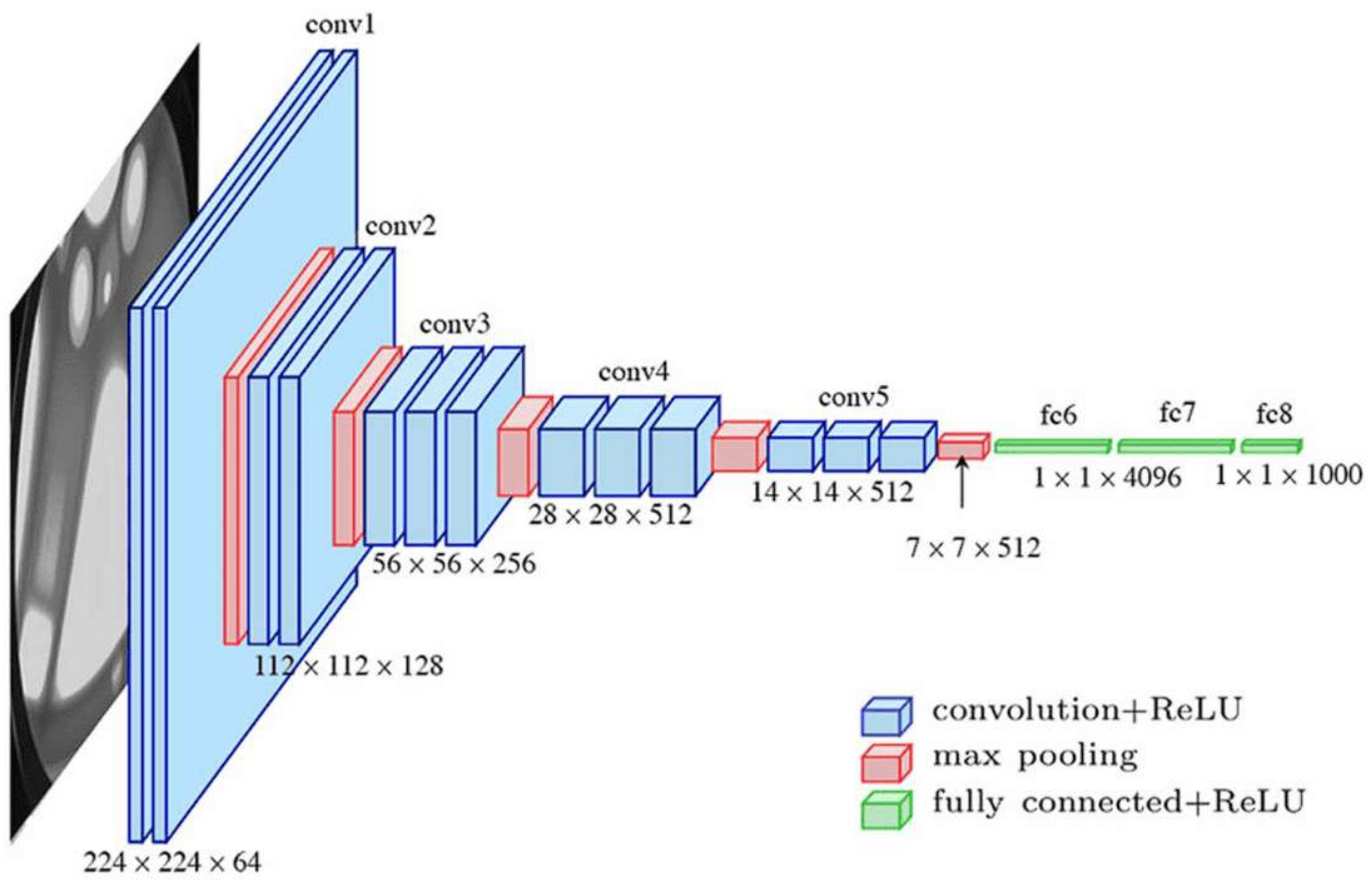
VGG

VERY DEEP CONVOLUTIONAL NETWORKS FOR LARGE-SCALE IMAGE RECOGNITION

Karen Simonyan* **& Andrew Zisserman**+
Visual Geometry Group, Department of Engineering Science, University of Oxford
{karen,az}@robots.ox.ac.uk



Resnet

by the number of stacked layers (depth). Recent evidence [41, 44] reveals that network depth is of crucial importance, and the leading results [41, 44, 13, 16] on the challenging ImageNet dataset [36] all exploit “very deep” [41] models, with a depth of sixteen [41] to thirty [16]. Many other non-

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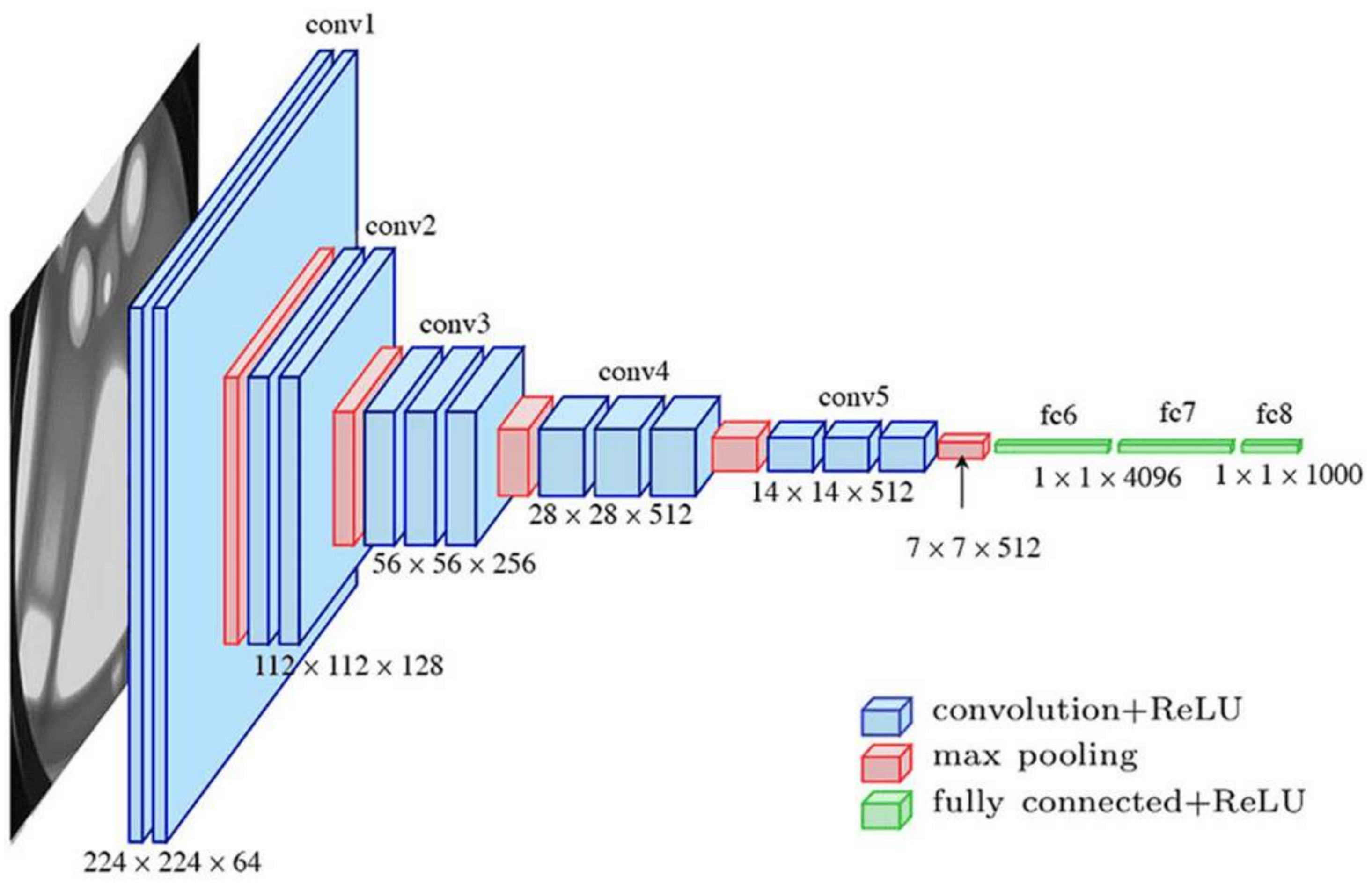

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Exponential gains in approximation capacity

The Power of Depth for Feedforward Neural Networks

Ronen Eldan
Weizmann Institute of Science
ronen.eldan@weizmann.ac.il

Ohad Shamir
Weizmann Institute of Science
ohad.shamir@weizmann.ac.il

Abstract

We show that there is a simple (approximately radial) function on $\mathbb{R}^d$, expressible by a small 3-layer feedforward neural networks, which cannot be approximated by any 2-layer network, to more than a certain constant accuracy, unless its width is exponential in the dimension. The result holds for virtually

Deep vs. Shallow Networks: an Approximation Theory Perspective

by

Hrushikesh N. Mhaskar[1] and Tomaso Poggio[2]
1. Department of Mathematics, California Institute of Technology, Pasadena, CA 91125
Institute of Mathematical Sciences, Claremont Graduate University, Claremont, CA 91711.
hrushikesh.mhaskar@cgu.edu
2. Center for Brains, Minds, and Machines, McGovern Institute for Brain Research, Massachusetts Institute of Technology, Cambridge, MA, 02139.
tp@mit.edu

What do we expect to show?

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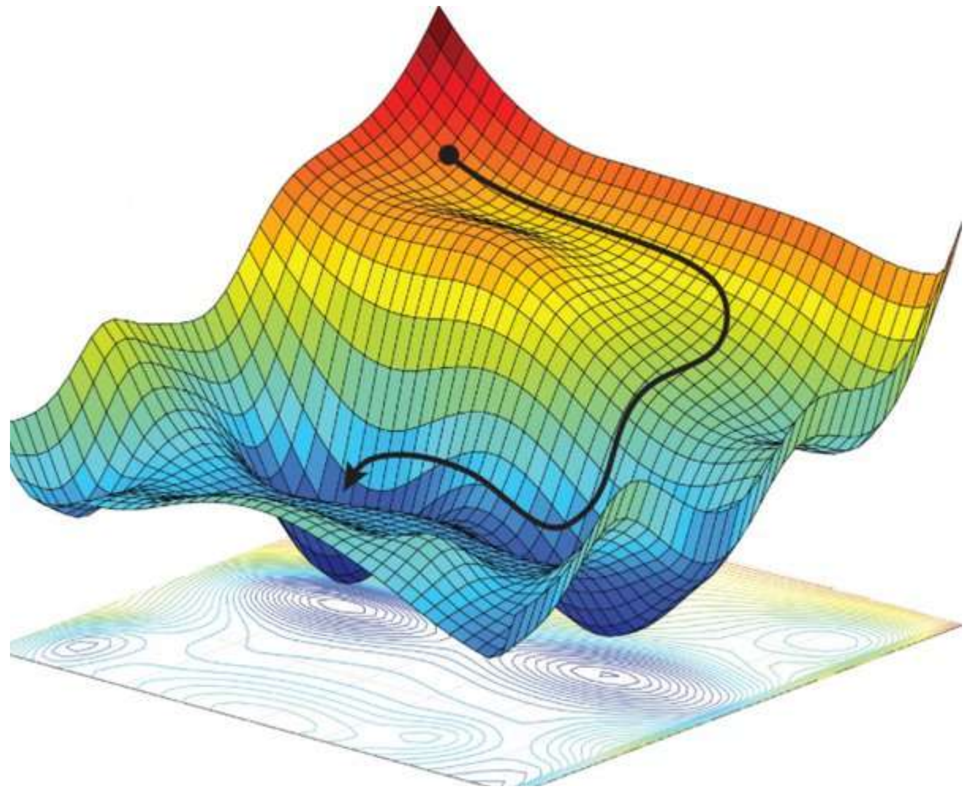
The quest for adaptivity

Posted on June 17, 2021 by Francis Bach



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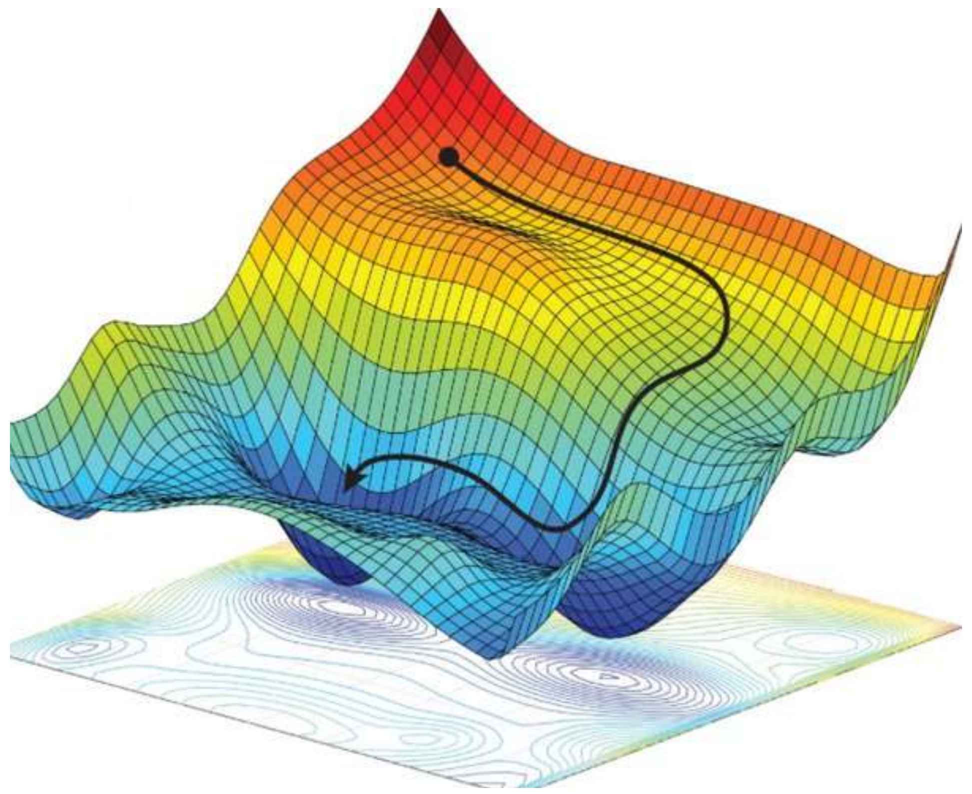

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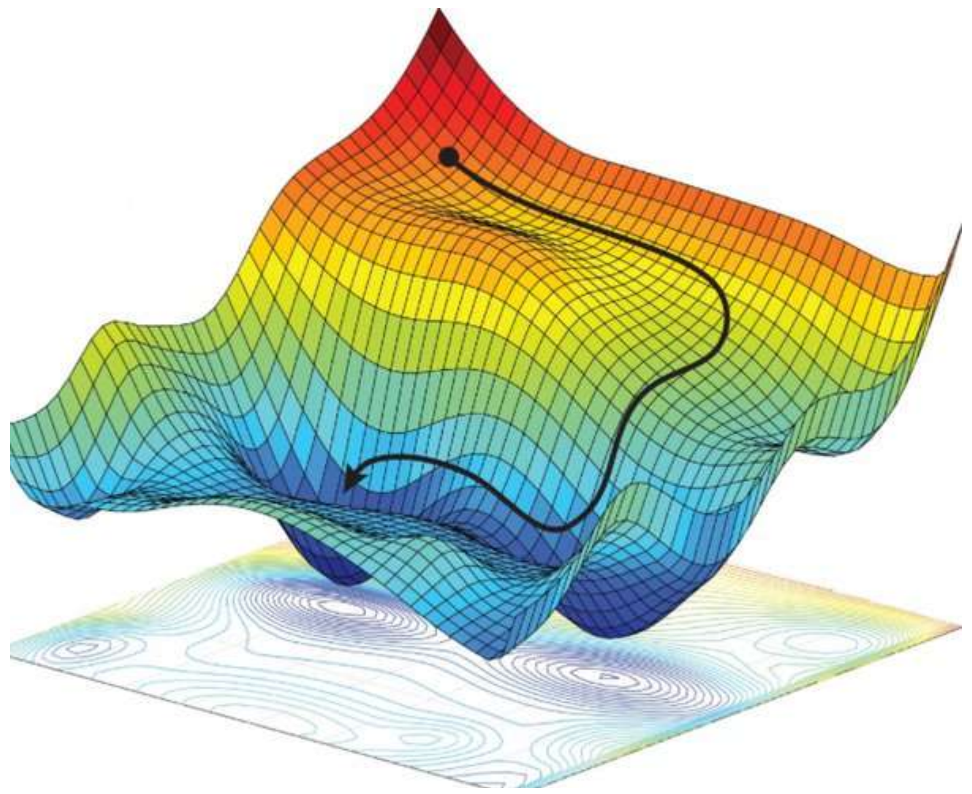

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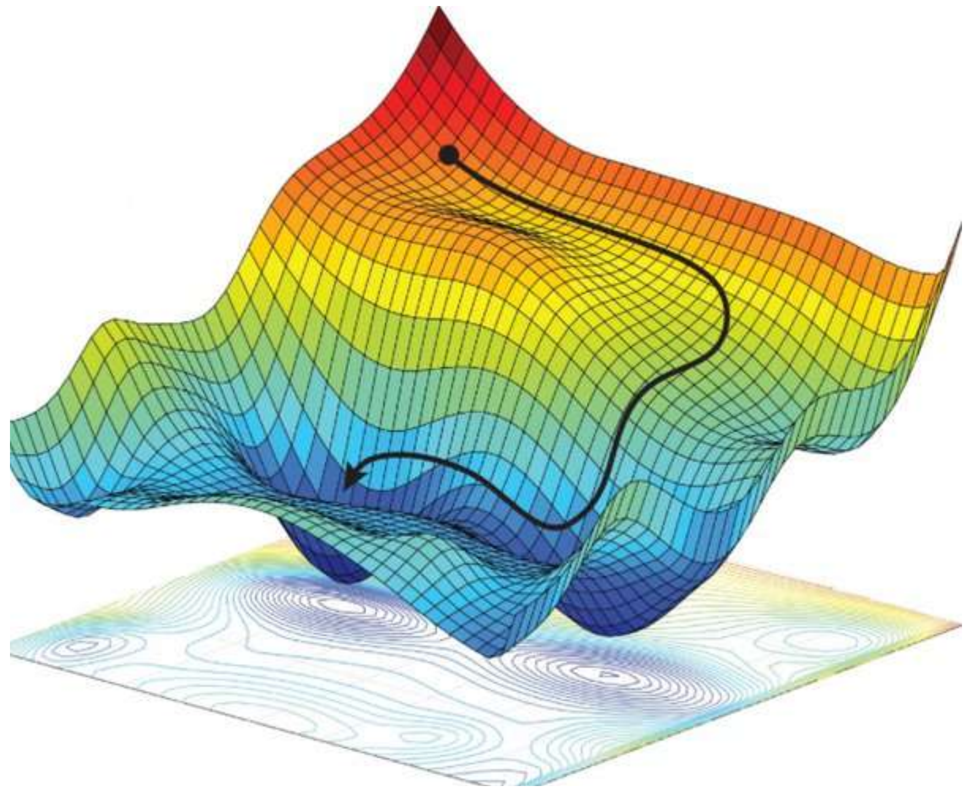
Lower-sample Complexity with adaptive algorithms



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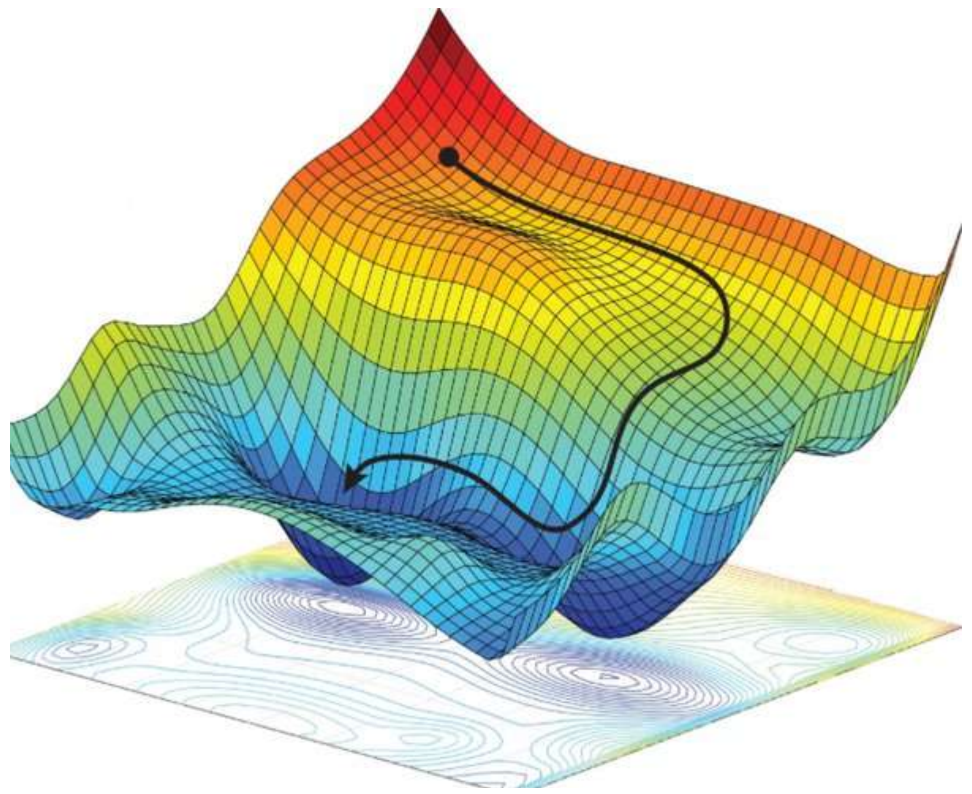

Gaussian data+ single, multi-index models

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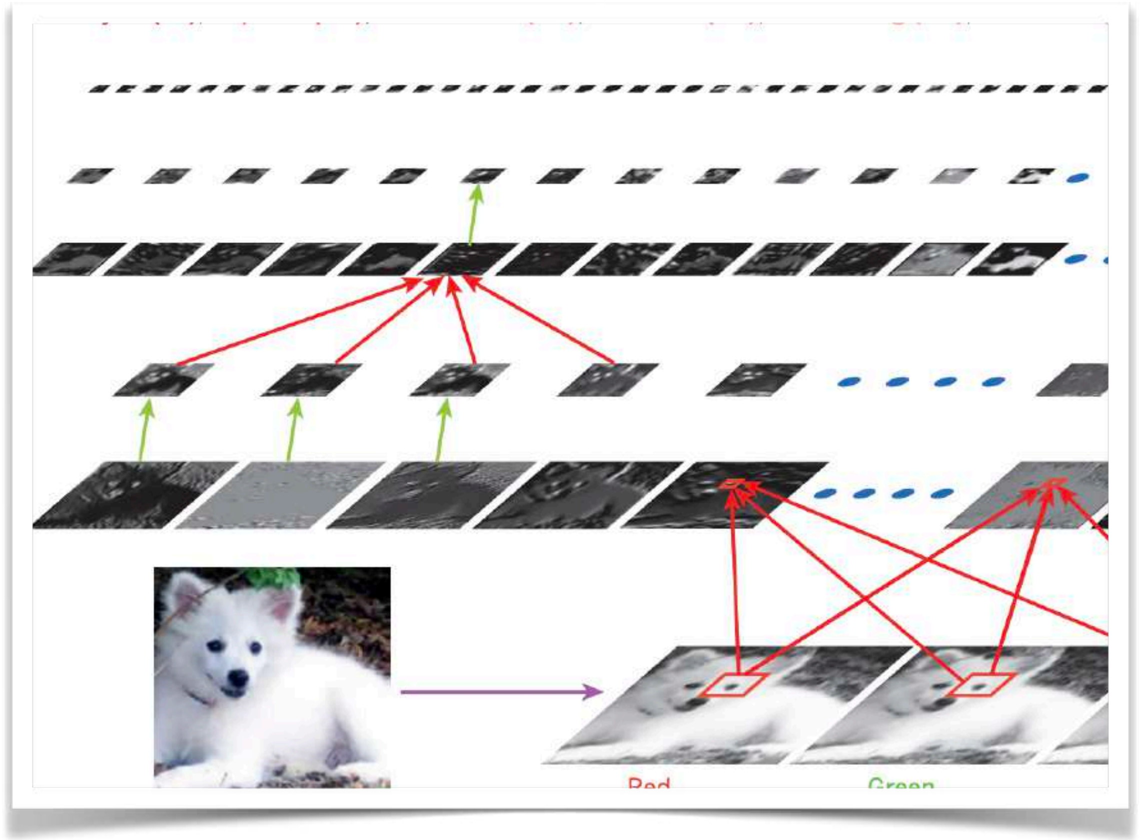

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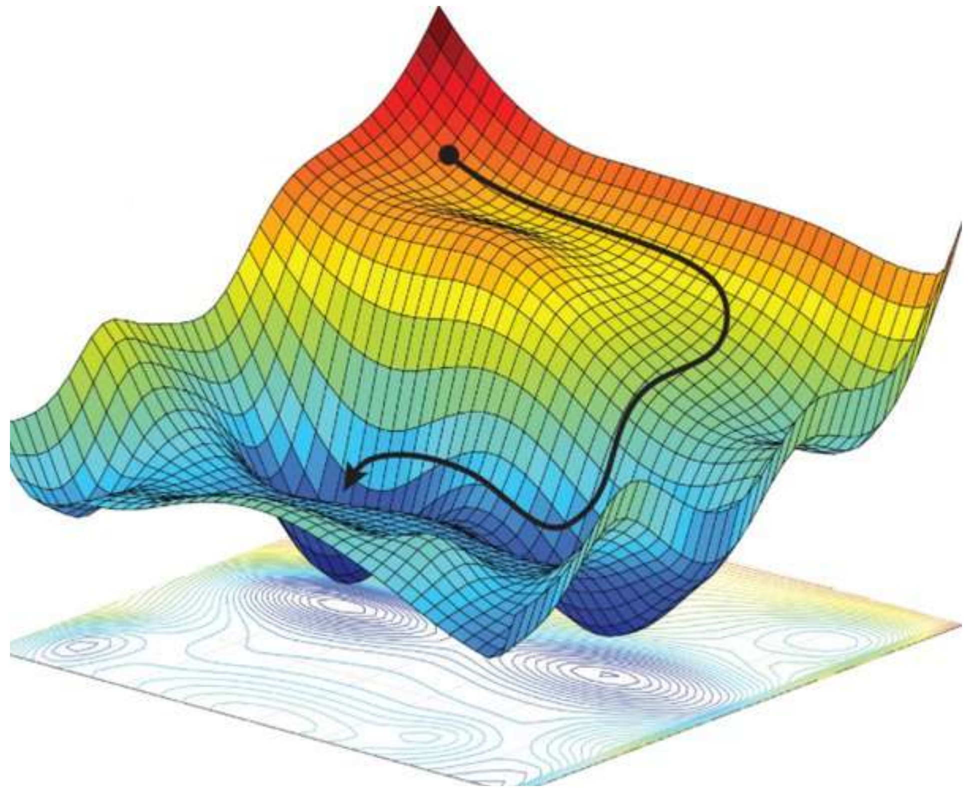

Hierarchical structure

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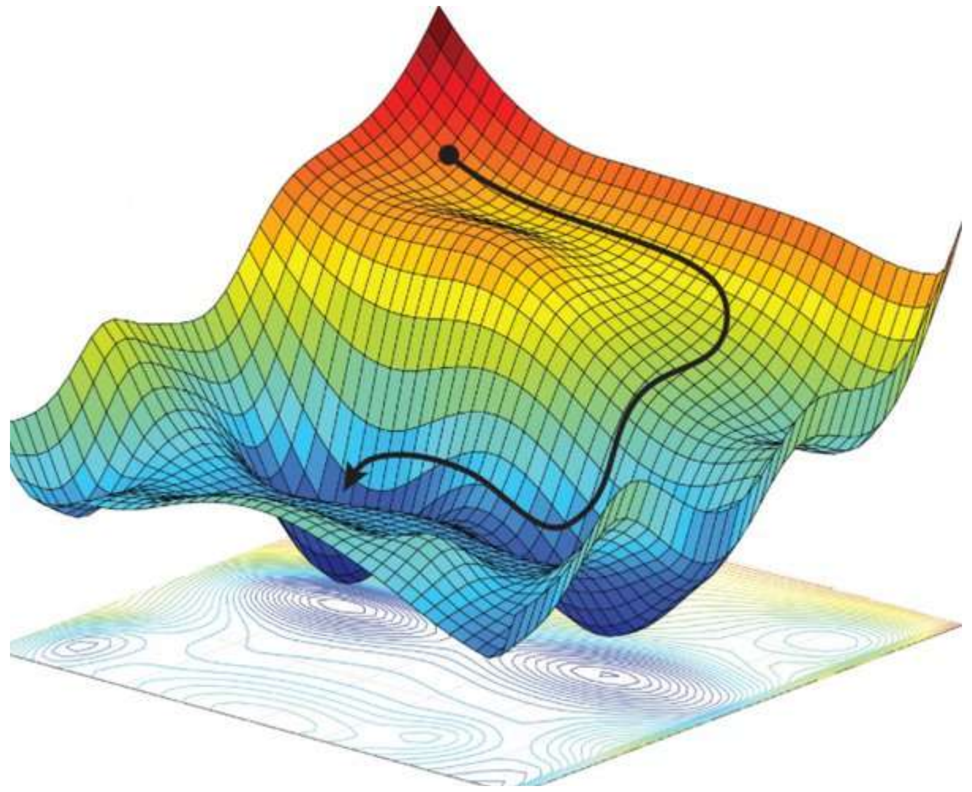

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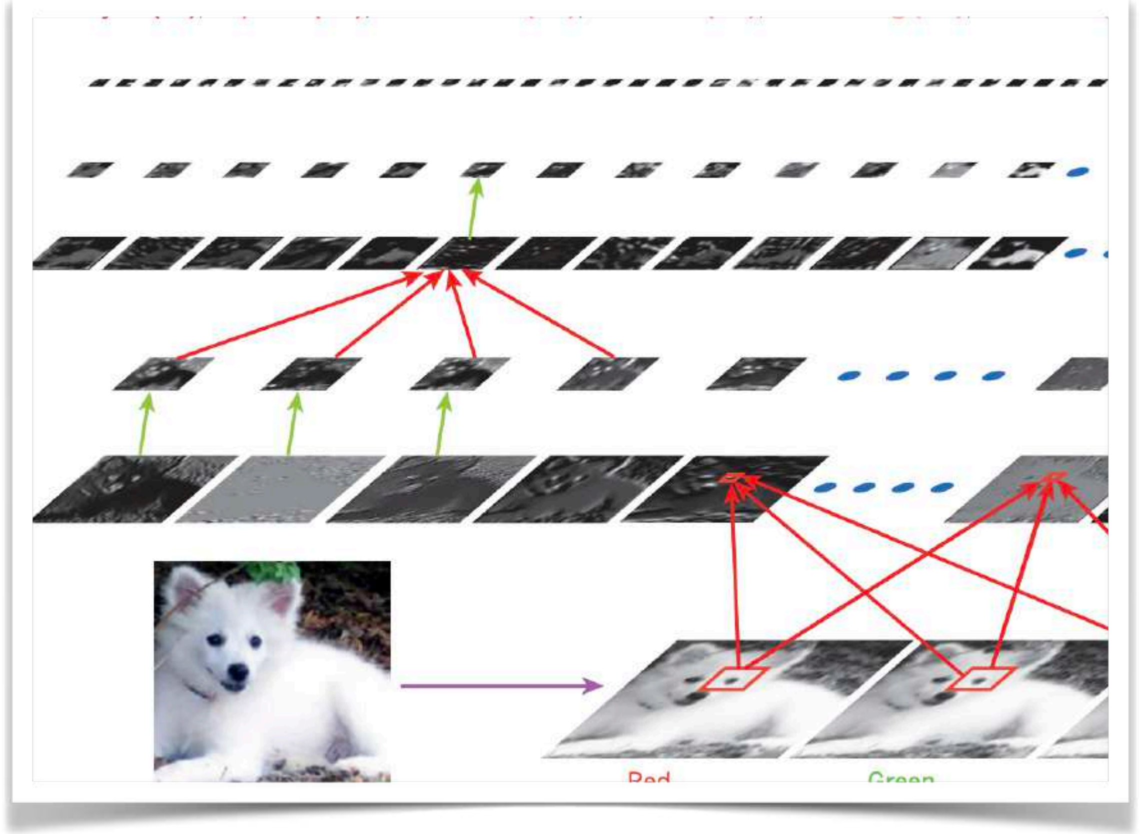

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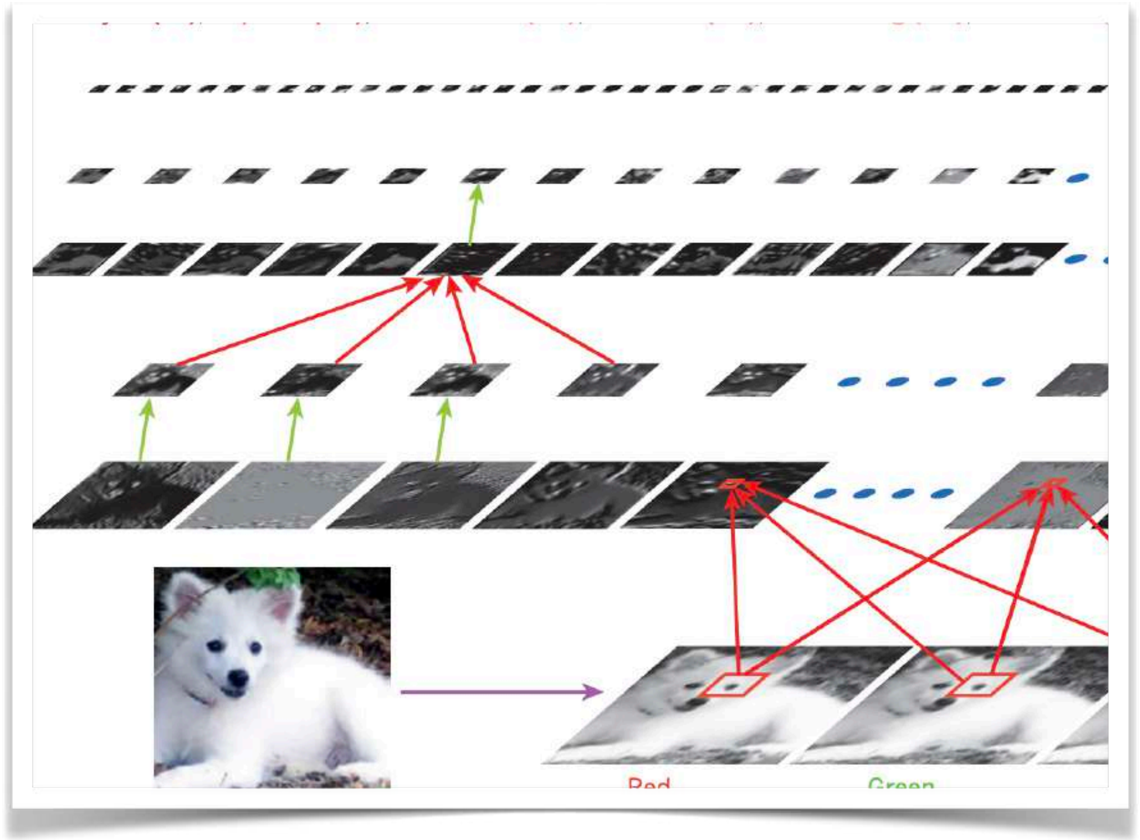

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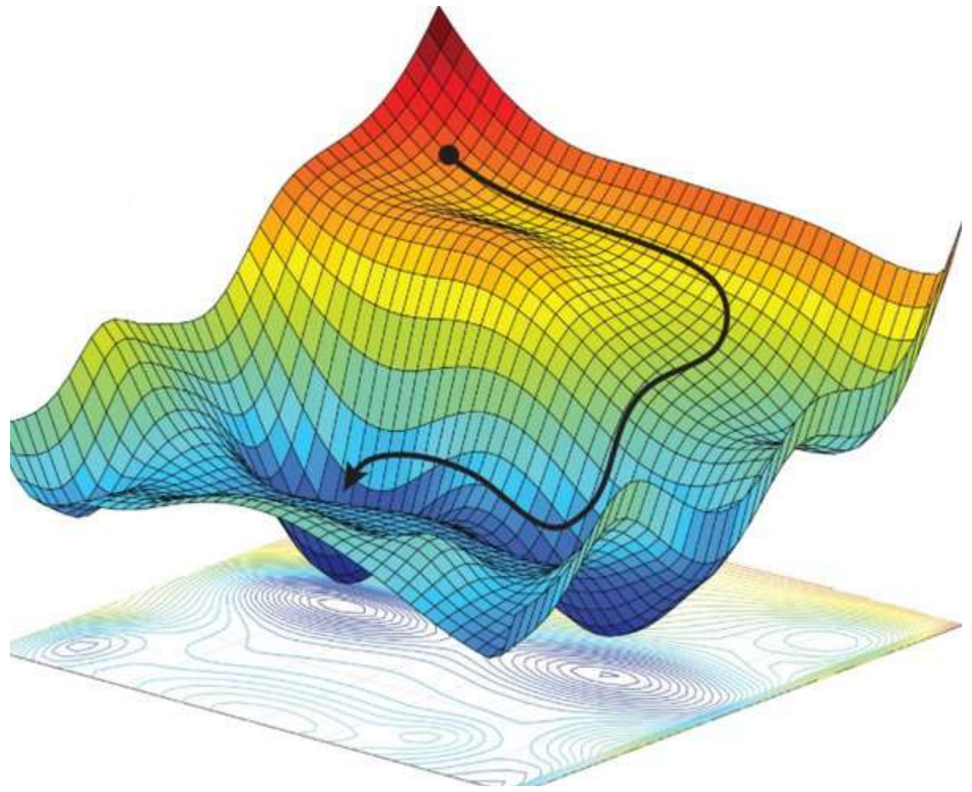

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Lower-sample Complexity With deep/multiple levels of adaptation.

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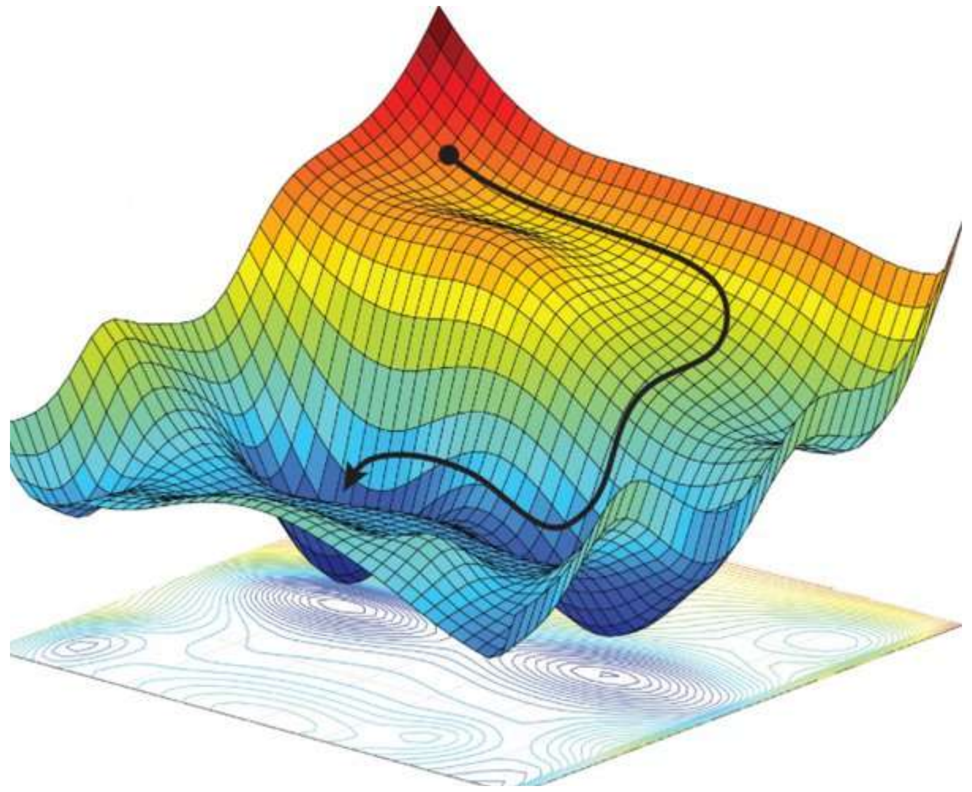

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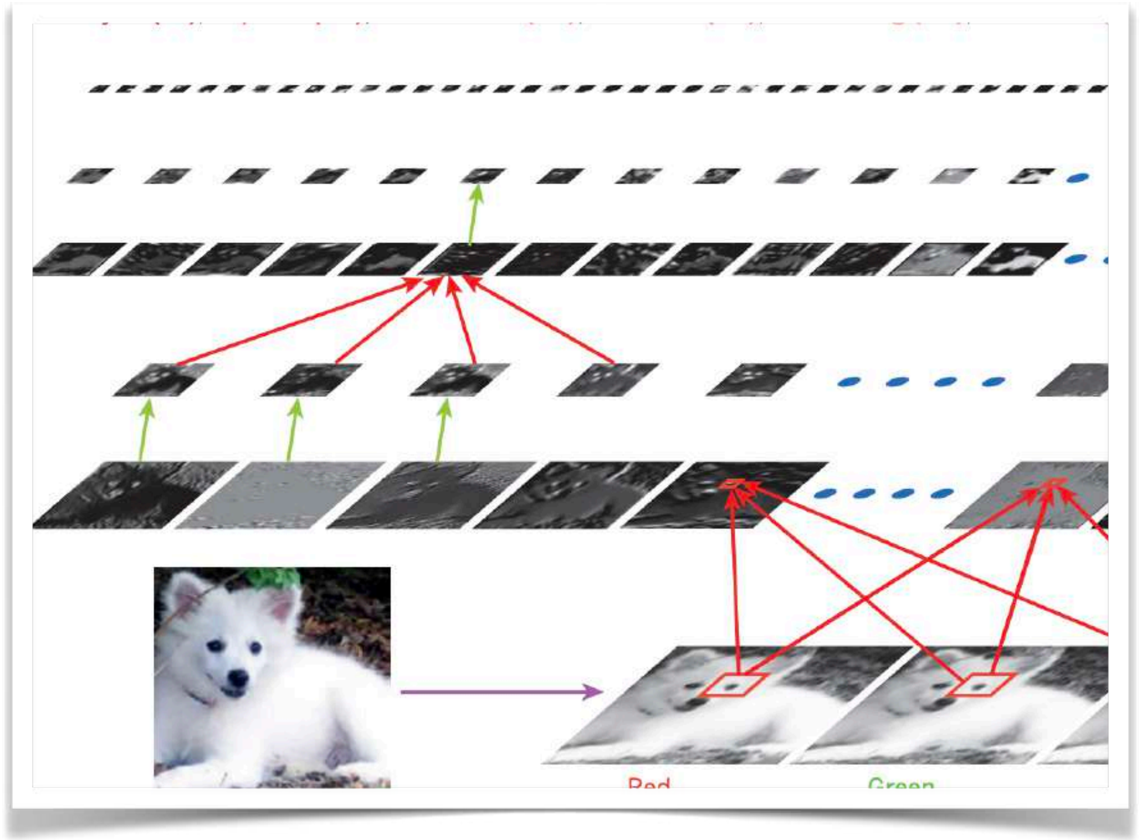

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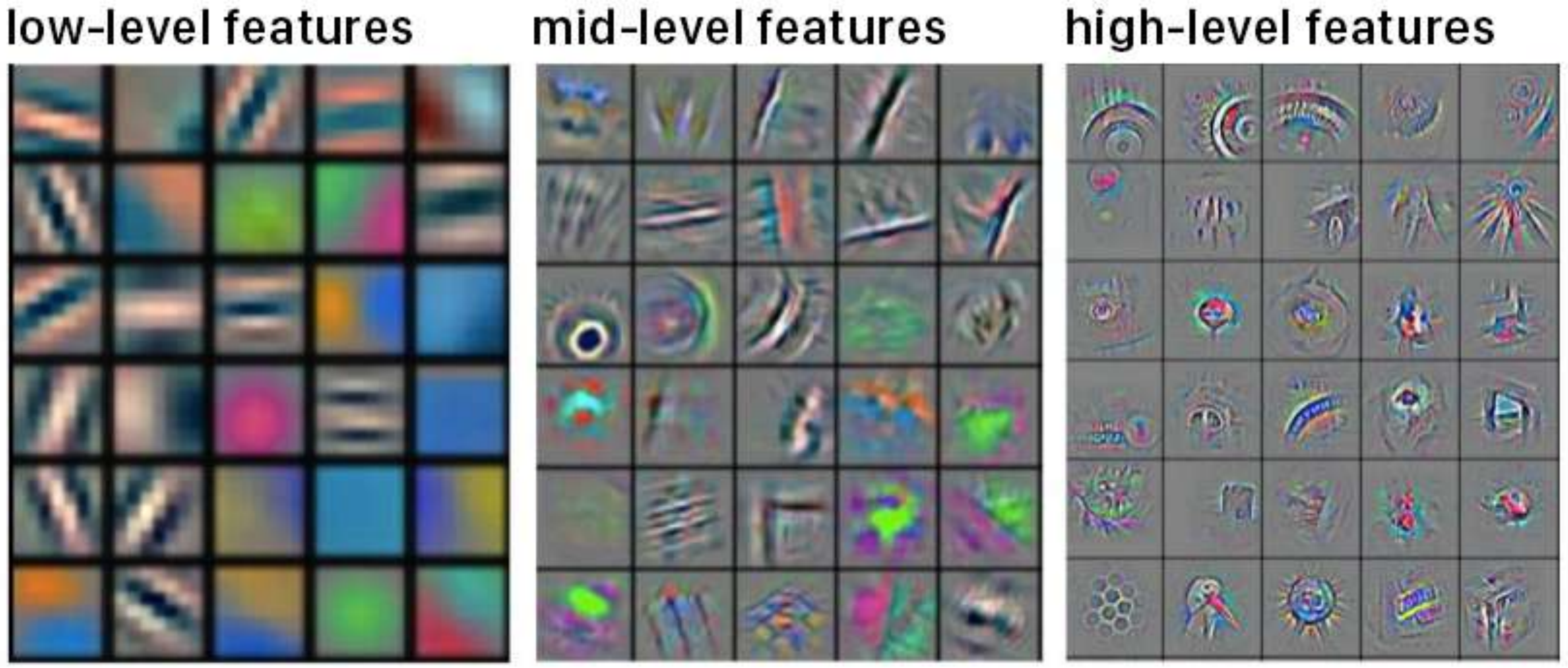

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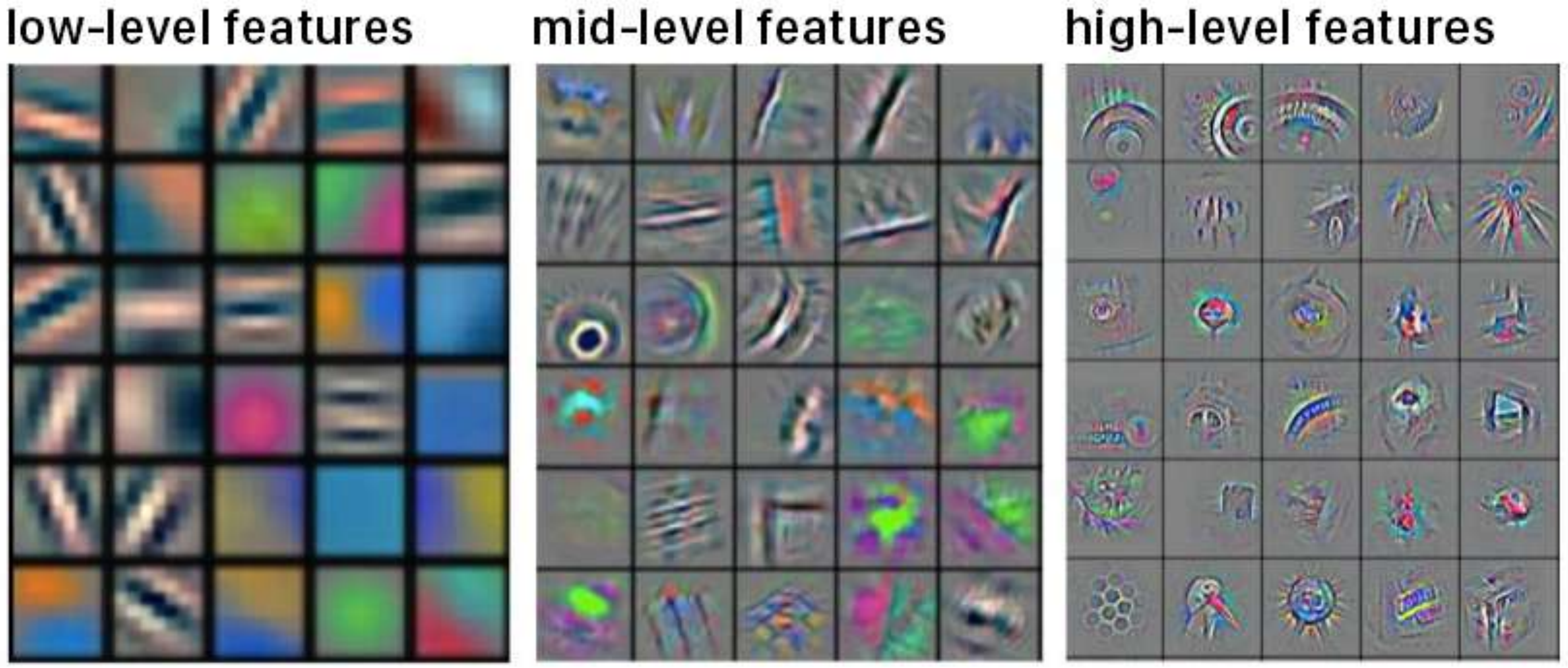

Large search space for target

Large effective dimension

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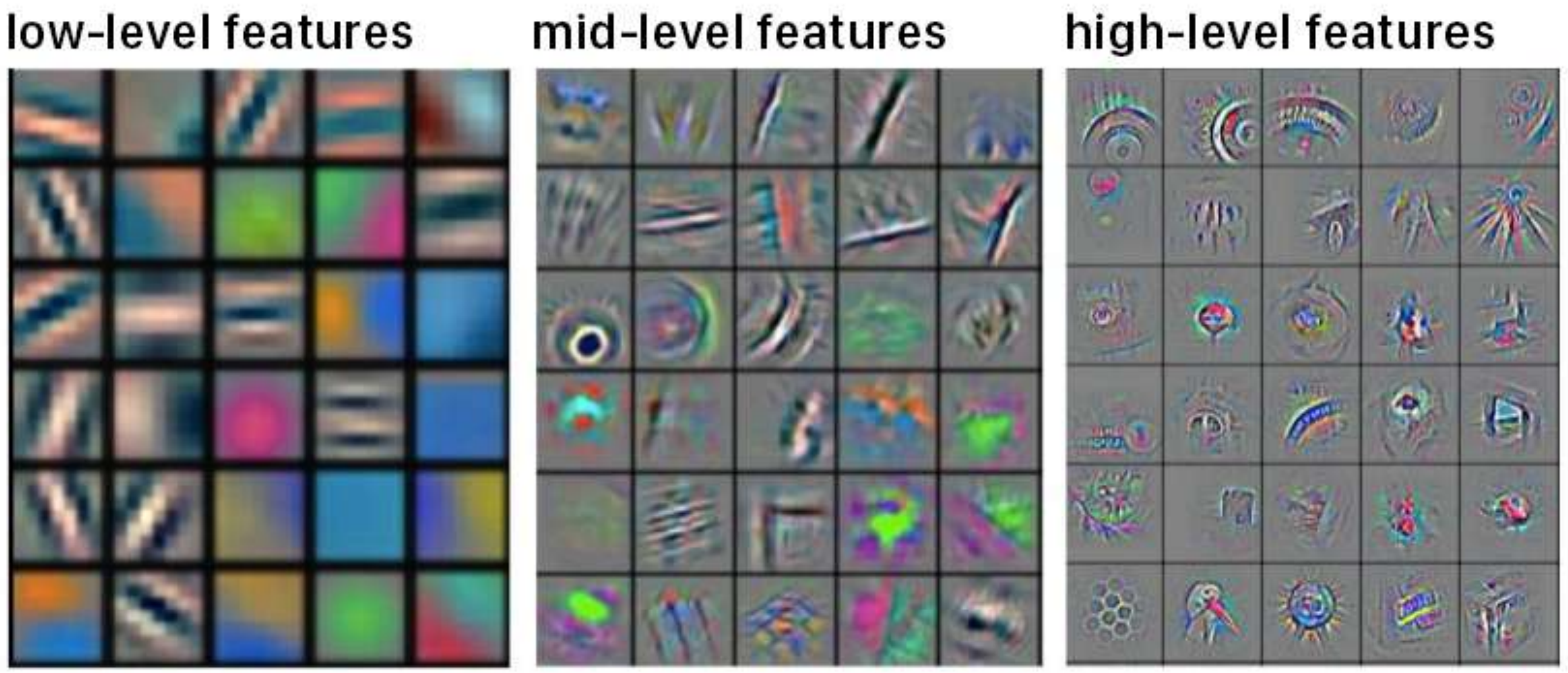

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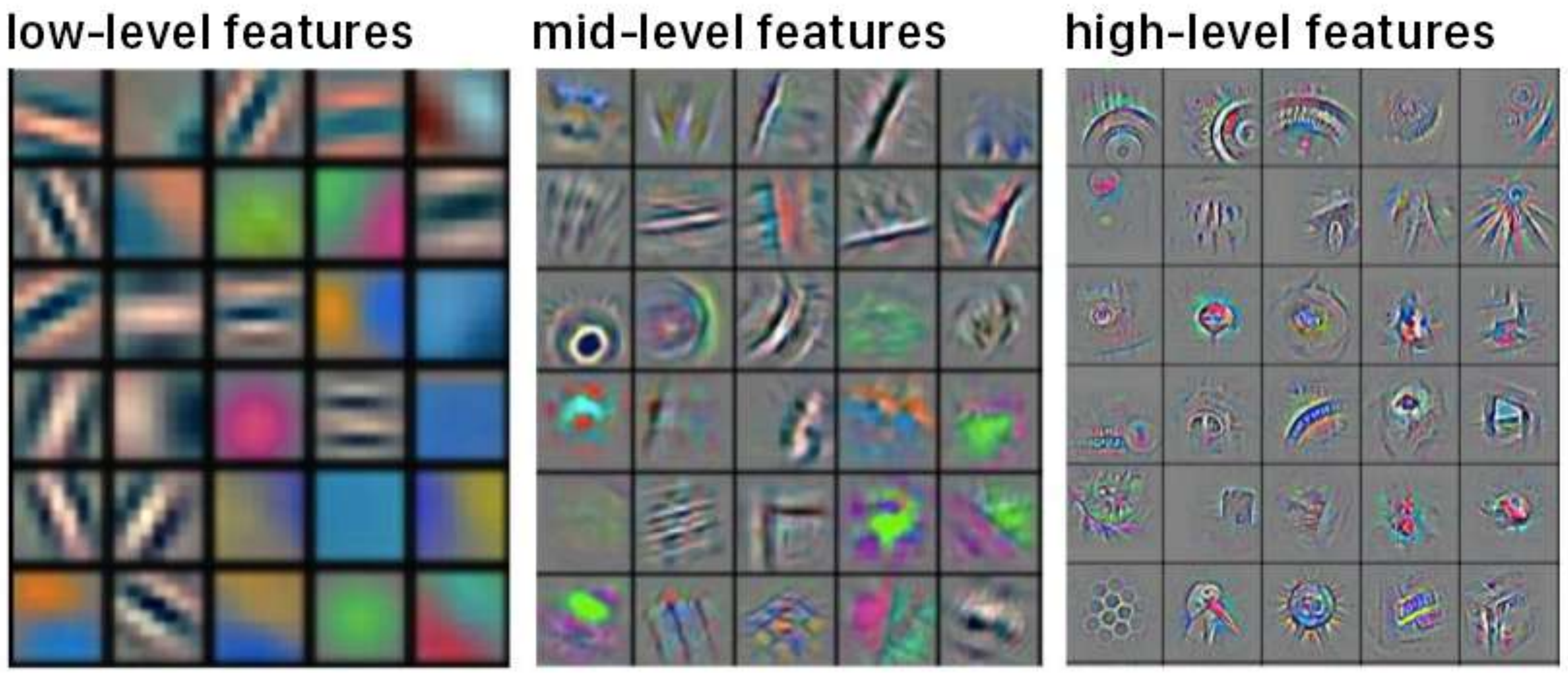

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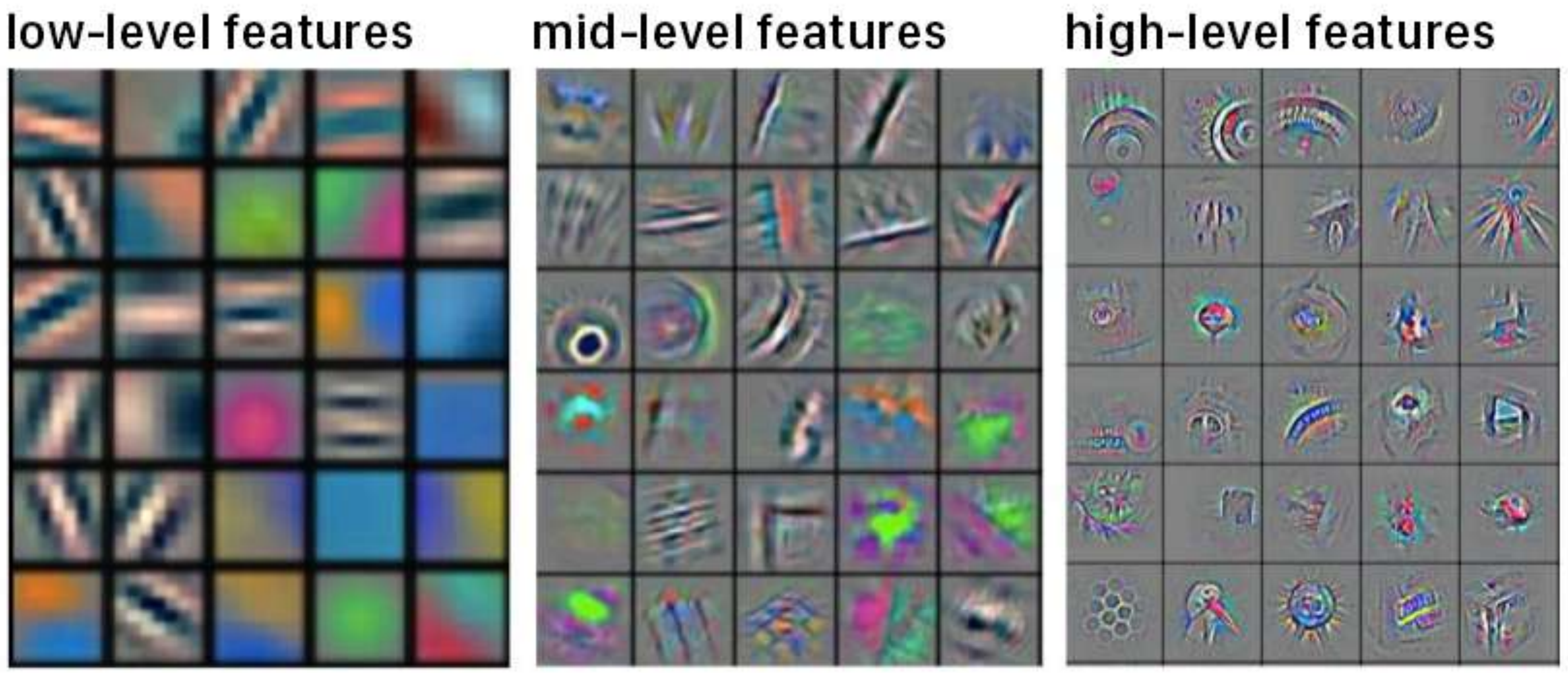

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Large effective dimension

Filtering of features of increasing complexity

small effective dimension

Akin to Coarse-graining/Renormalization (Gulio's talk)

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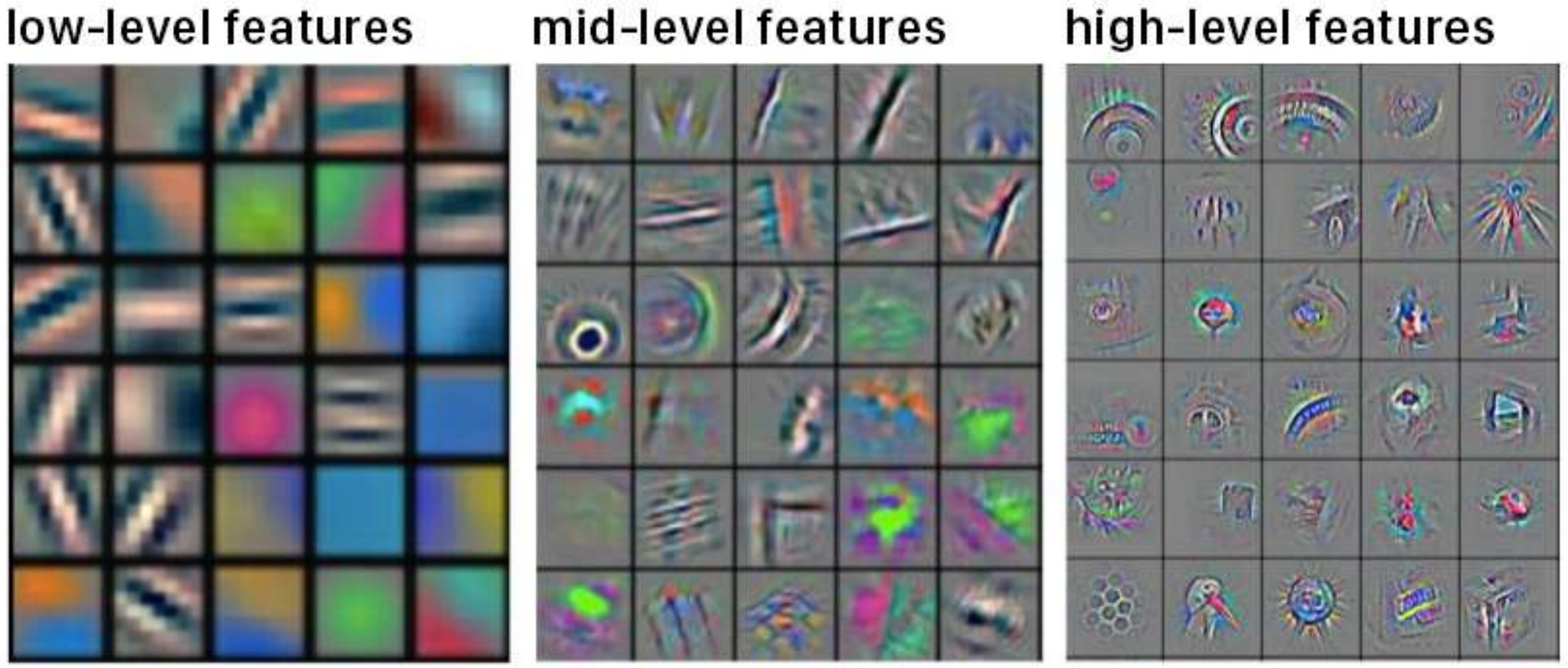
See also Cagnetta et al. 2025, Mossel et al., 2016, Bruna & Mallat 2013



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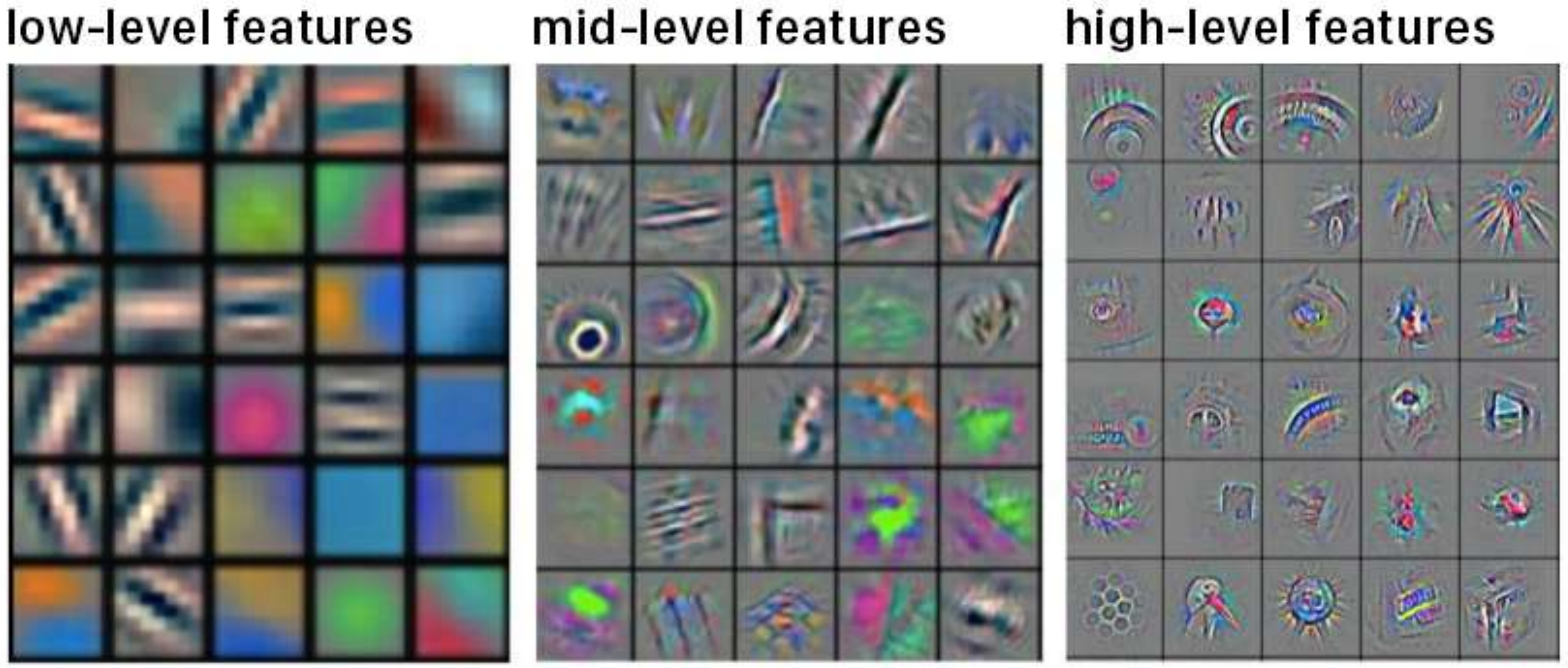

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Can we understand this in some analyzable setting?

I

Recap of two-layer NNs for AGI

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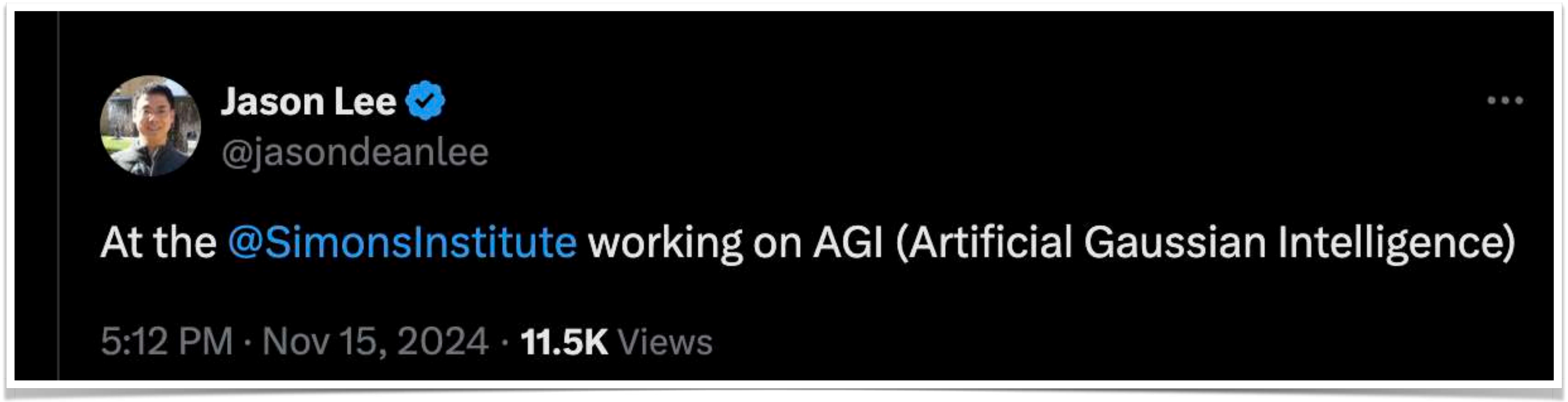
Recap of two-layer NNs for AGI



Neural Networks without Feature Learning

Random features

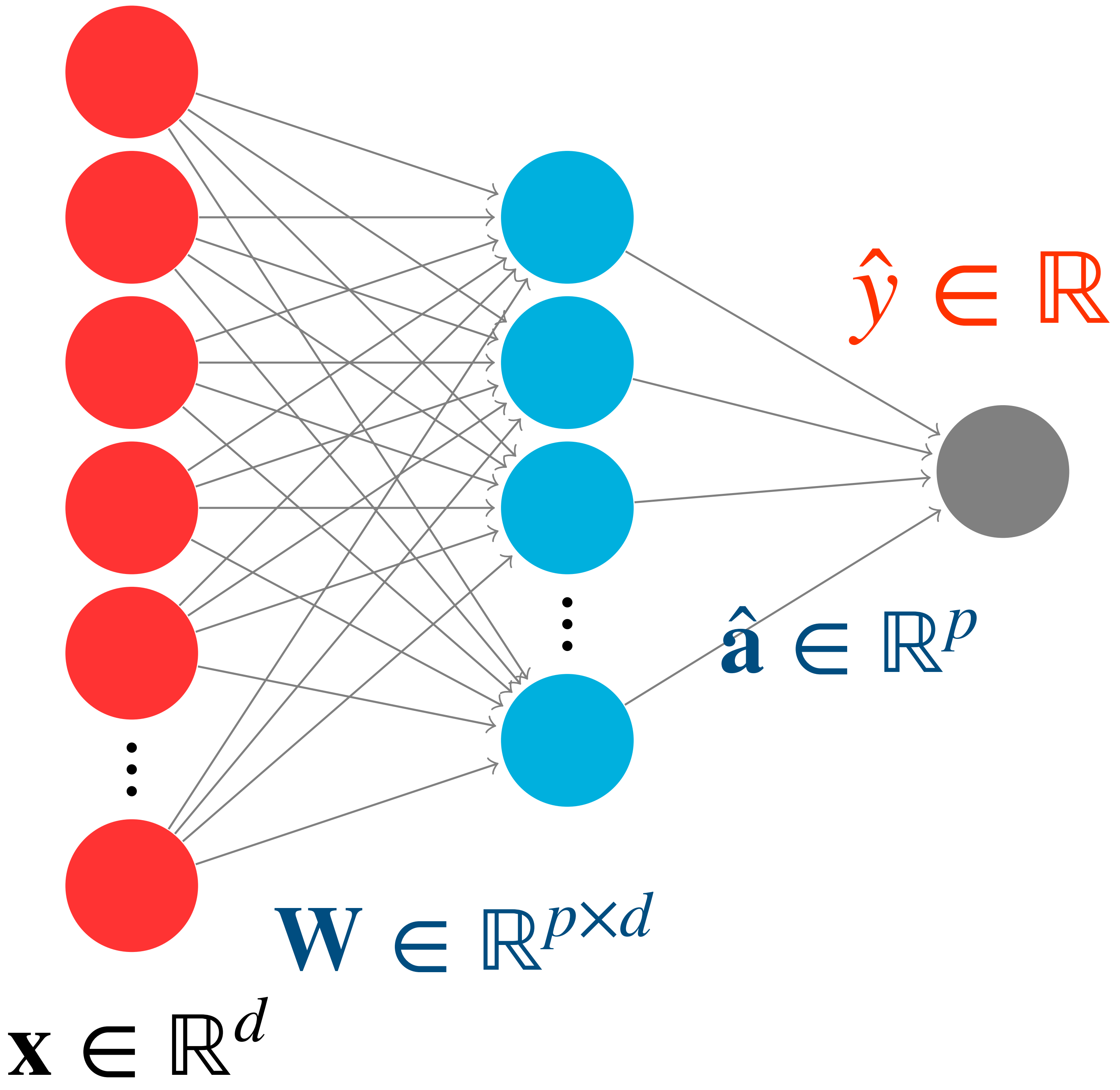
[Balcan,Blum, Vempala ’06, Rahimi-Recht ’17…]

$$\hat{y} = \hat{f}(\mathbf{x}) = \hat{\mathbf{a}} \cdot \sigma(W\mathbf{x})$$

No training of the first layer: W is fixed

$$\hat{y} = \hat{f}(\mathbf{x}) = \sum_{i=1}^{p} \hat{a}_i \sigma_i(\langle \mathbf{w}_i, \mathbf{x} \rangle) = \sum_{i=1}^{p} \hat{a}_i \Phi_{\mathrm{CK}}(\mathbf{x})$$

Computationally easy (linear regression)



Very popular setting among theoreticians

Equivalent to Neural Tangent Kernel/Lazy Regime/Kernel methods/ etc..

[Jacot, Gabriel, Hongler ’18; Lee, Jaehoon, et al. 18; Chizat, Bach ’19,…]

Recap of Sample Complexity

Theorem (Informal) [Mei, Misiakiewicz, Montanari '22]

In *absence* of feature learning (i.e. *at initialization)* one can only learn a ***polynomial*** **approximation of $f^\star$ of degree κ with** $\min(n,p) = O(d^\kappa)$

$$f^\star(\mathbf{x}) = \mathrm{cst} + \sum_i \mu_i^{(1)} h_i^\star + \sum_{ij} \mu_{ij}^{(2)} h_i^\star h_j^\star + \sum_{ijk} \mu_{ijk}^{(3)} h_i^\star h_j^\star h_k^\star + \dots$$

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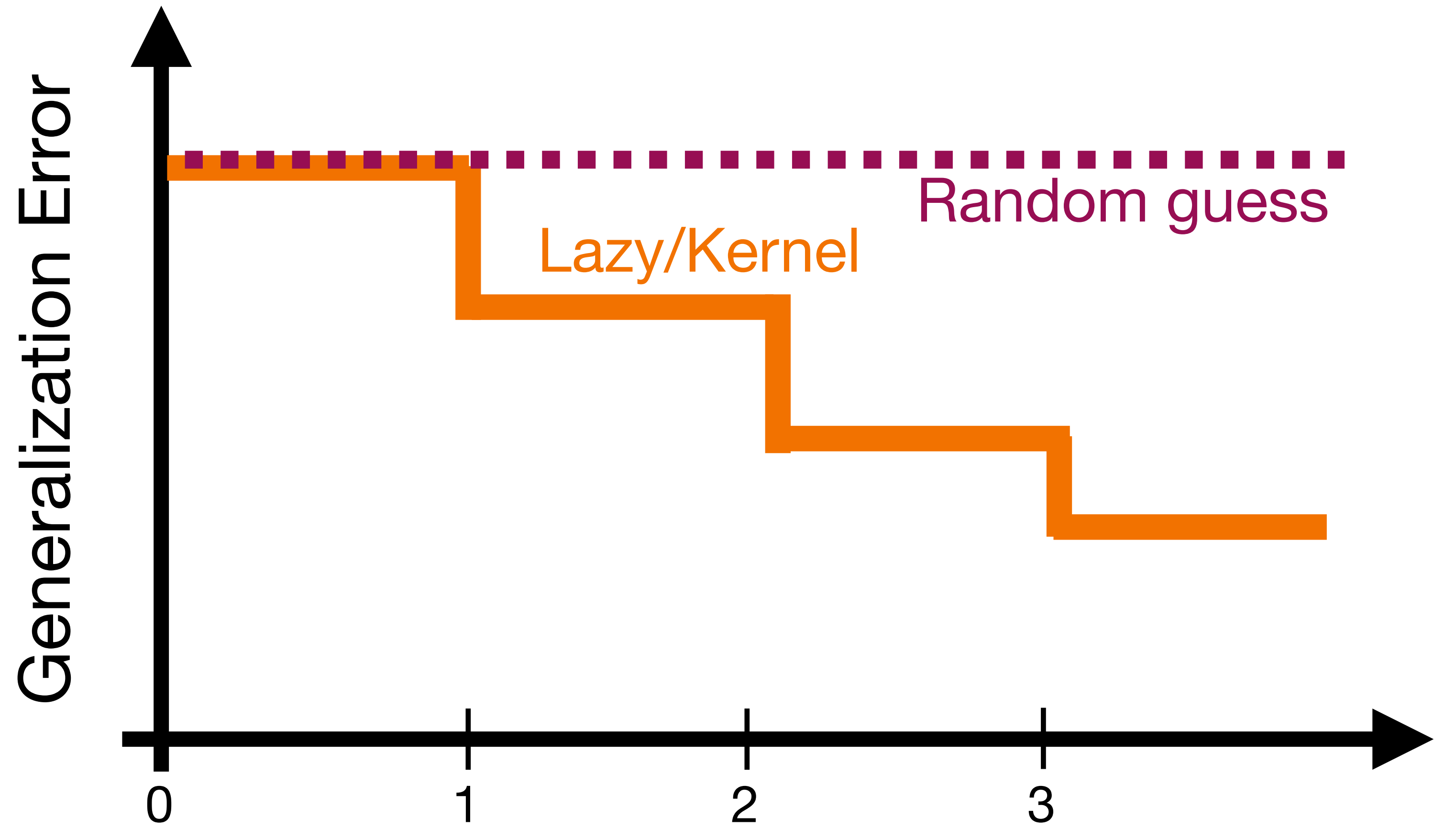

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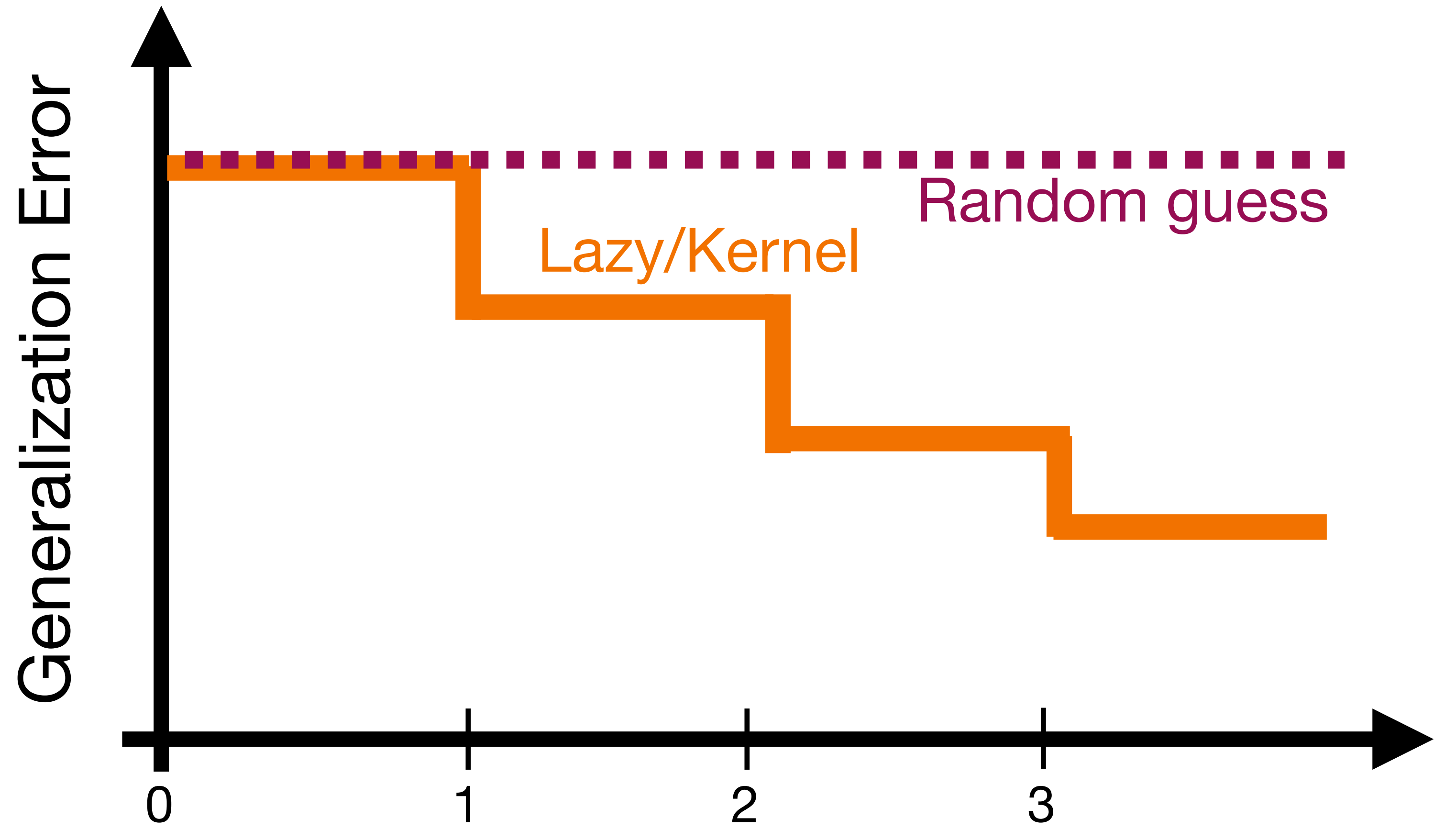

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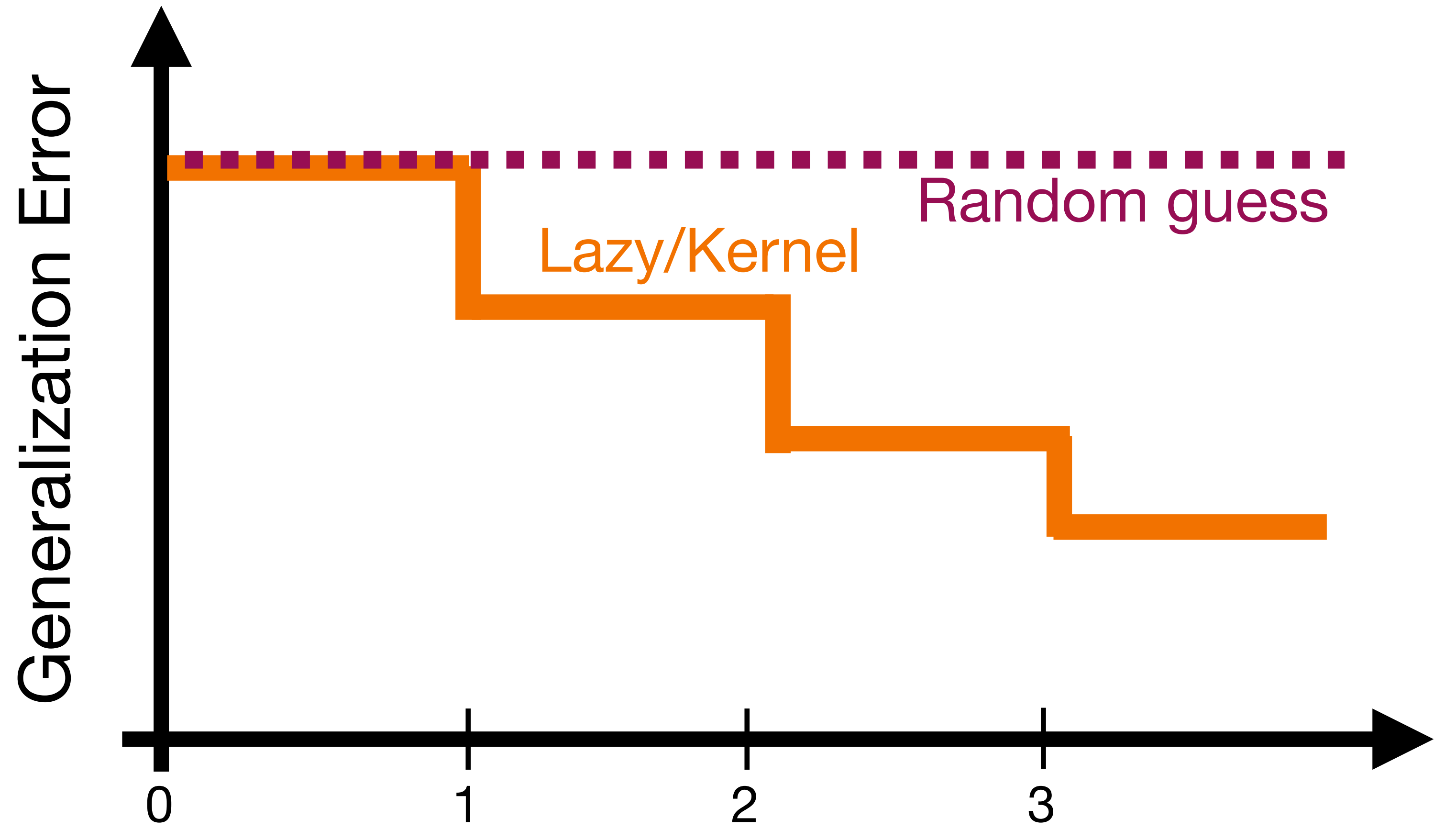

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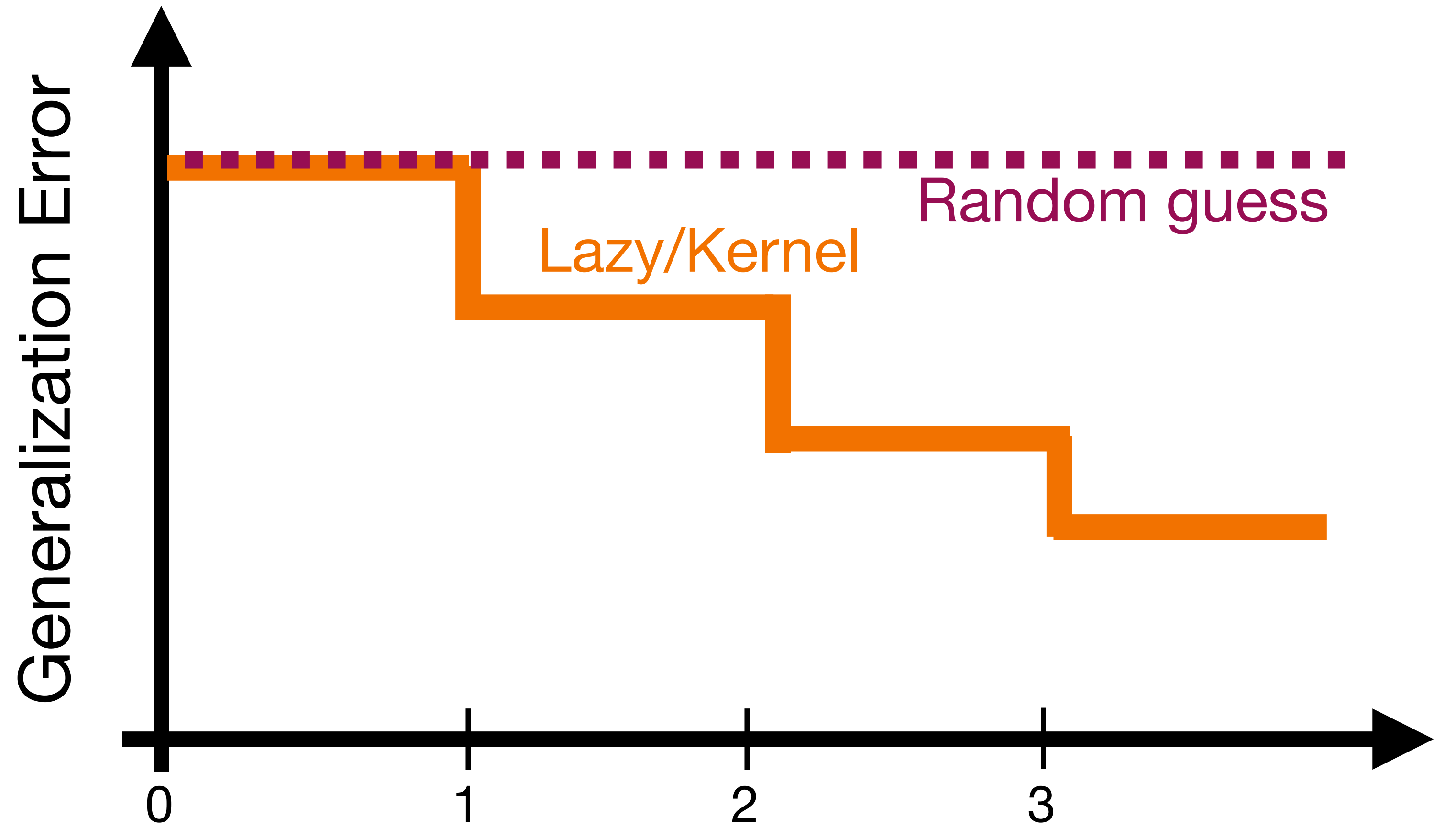

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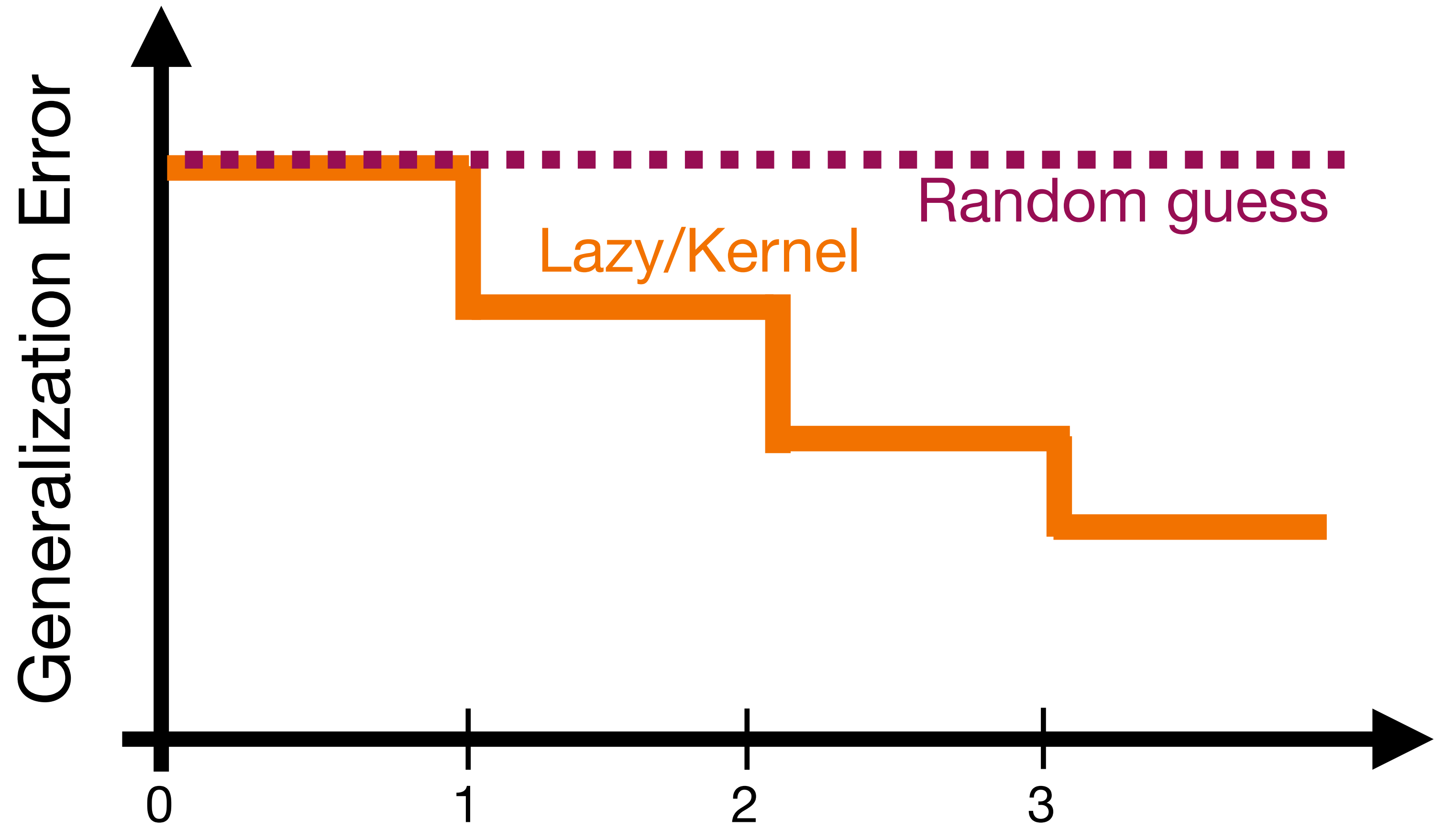
No adaptivity $\Longrightarrow$ searching in a $\mathcal{O}(d^k)$ dimensional subspace of polynomials*



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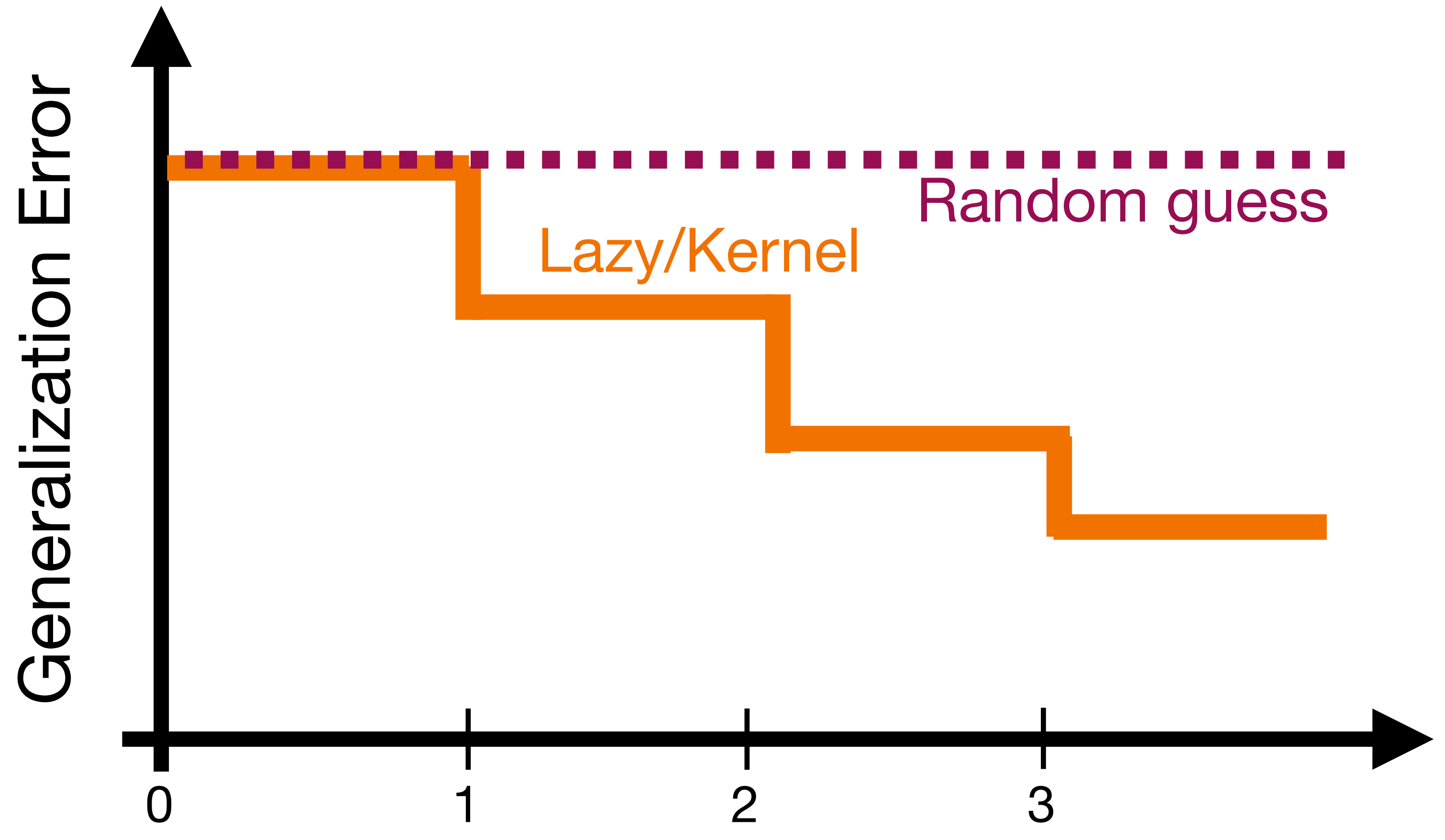

No adaptivity $\Longrightarrow$ searching in a $\mathcal{O}(d^k)$ dimensional subspace of polynomials*

***possibly excluding a few "special" polynomials.**

See also [El Karaoui '10; Mei-Montanari '19; Gerace '20; Jacot, Simsek, Spadaro, Hongler, Gabriel '20; Hu, Lu, '20; Dhifallah, Lu '20; Loureiro, Gerbelot, Cui, Goldt '21; Montanari & Saeed '22; Xiao, Hu, Misiakiewicz, Lu, Pennington '22; Dandi '23; Aguirre-López, Franz, Pastore '24]

Advantage of Feature Learning

Ben Arous et al. 2021, Abbe et al. 2022,2023, Damian et al. 2022, Dandi et al. 2023, 2024, etc.

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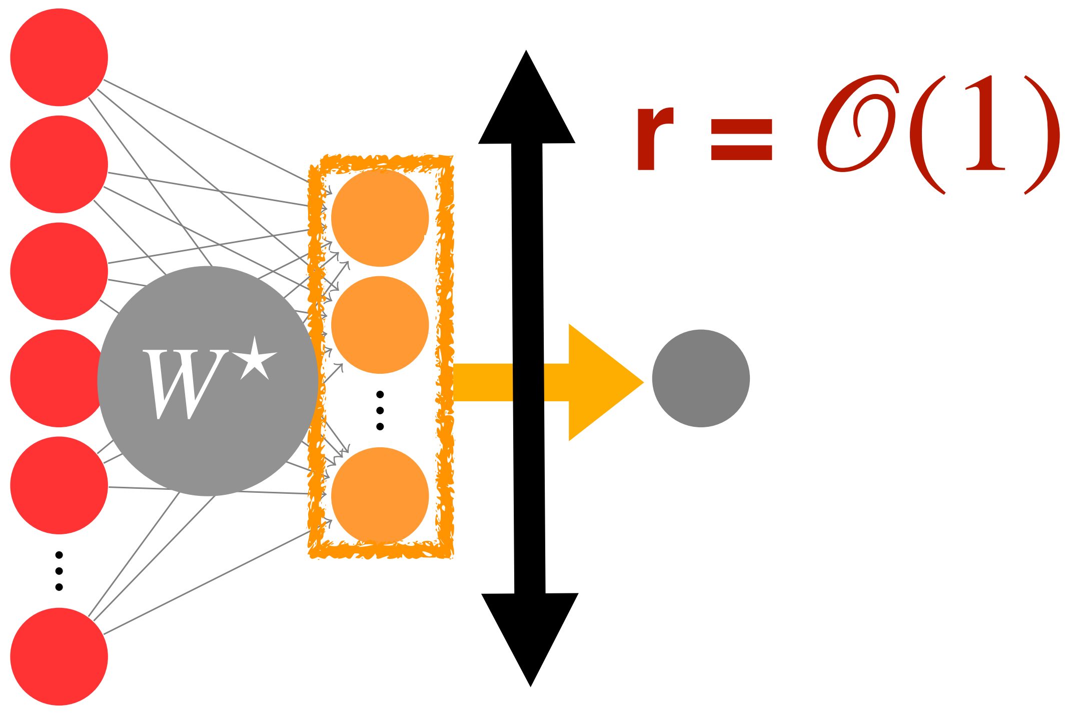

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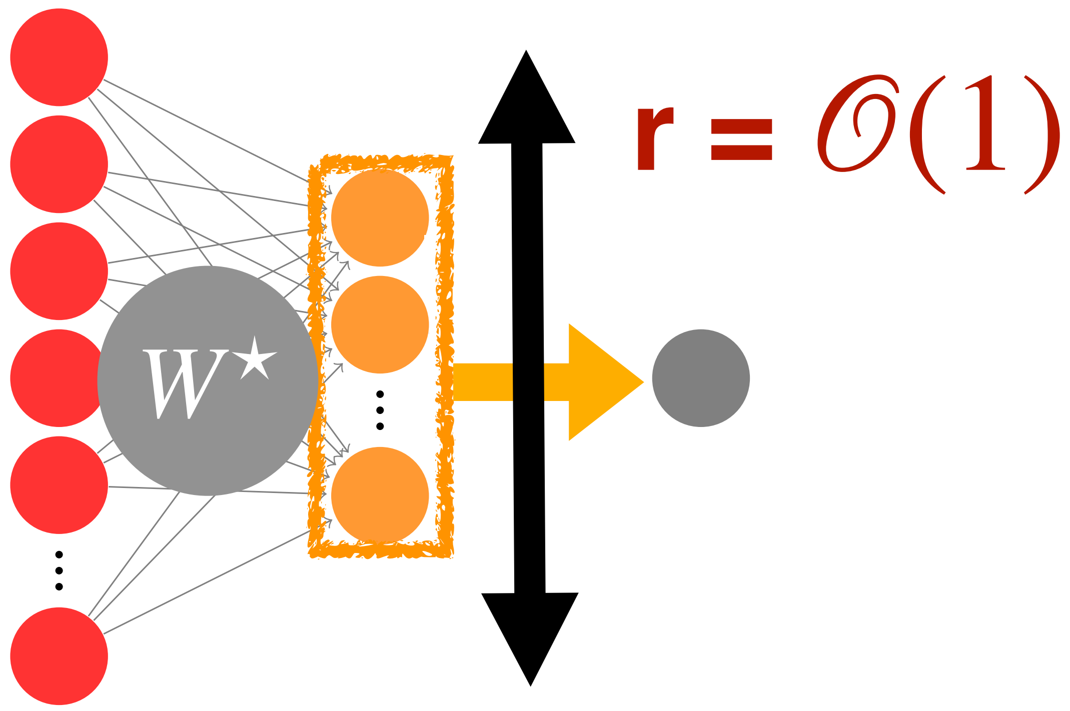

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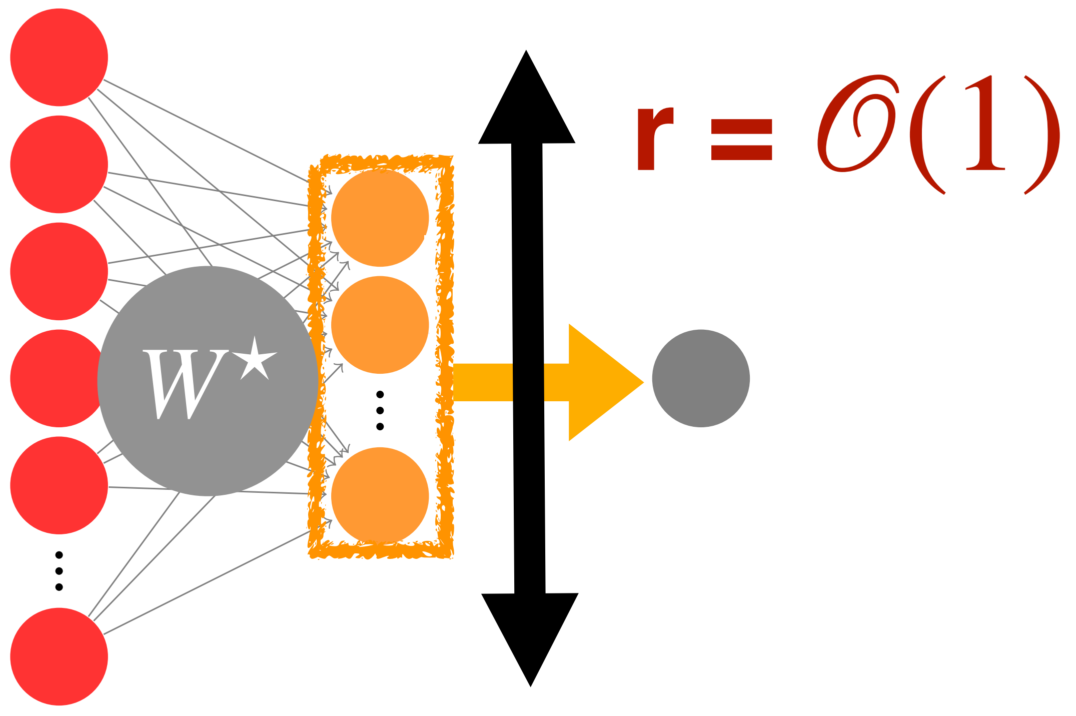

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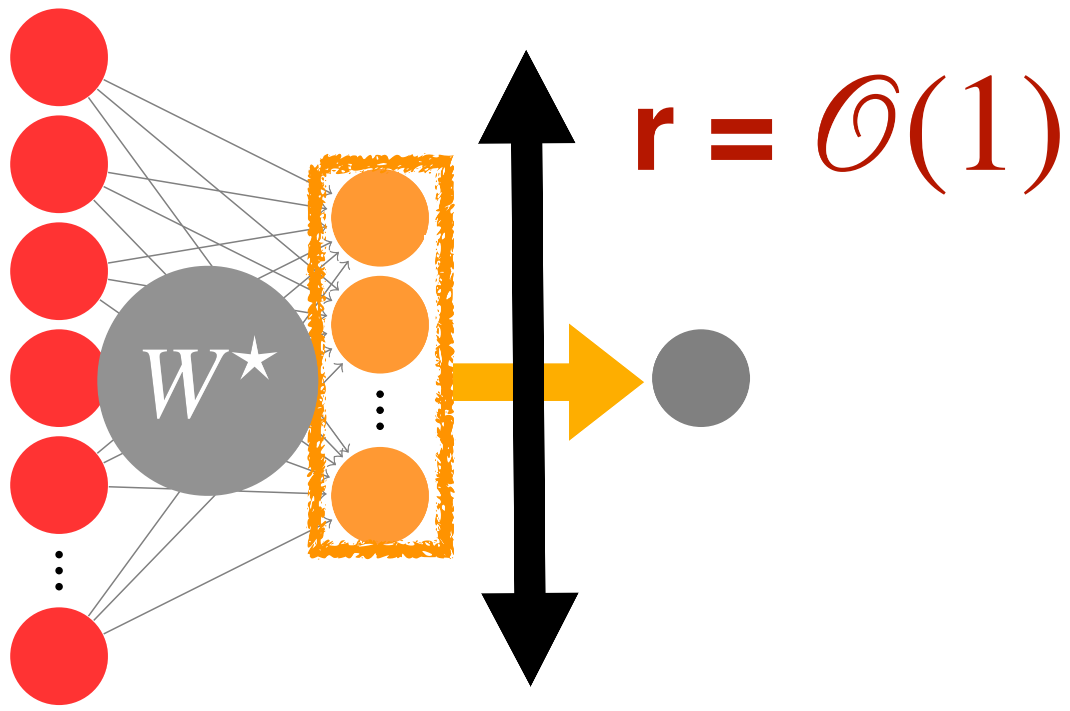

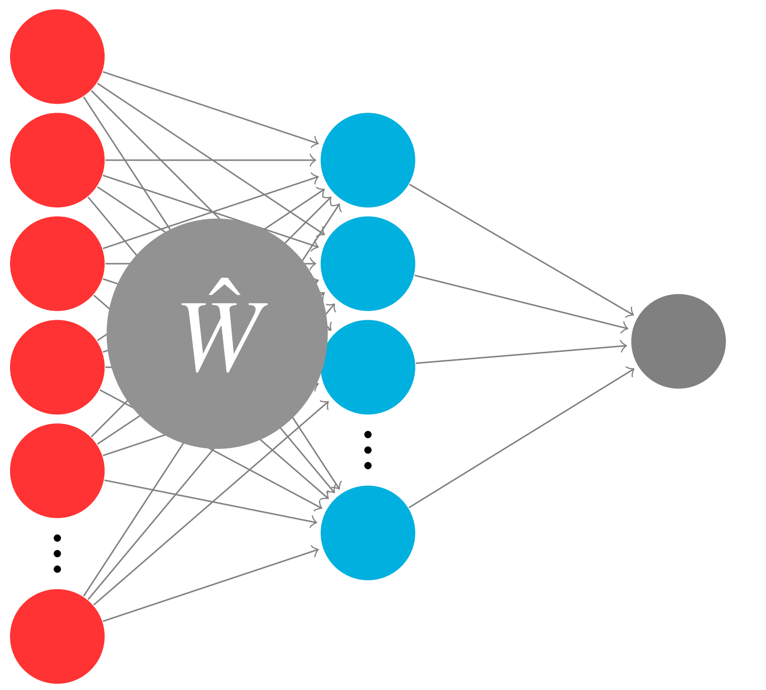

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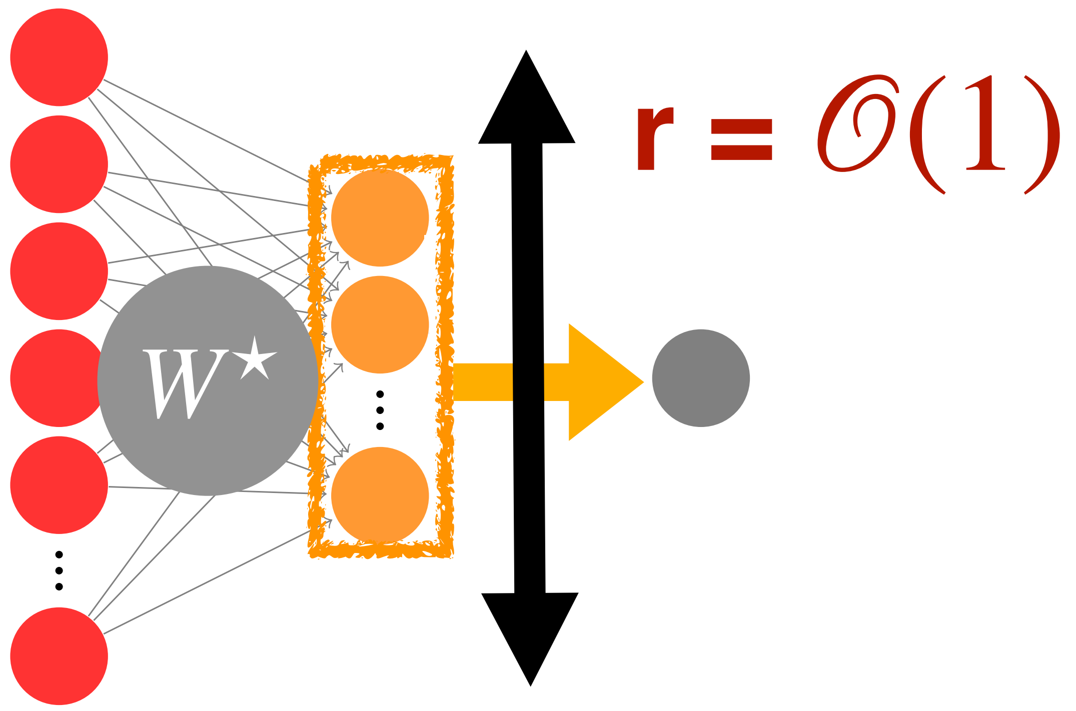

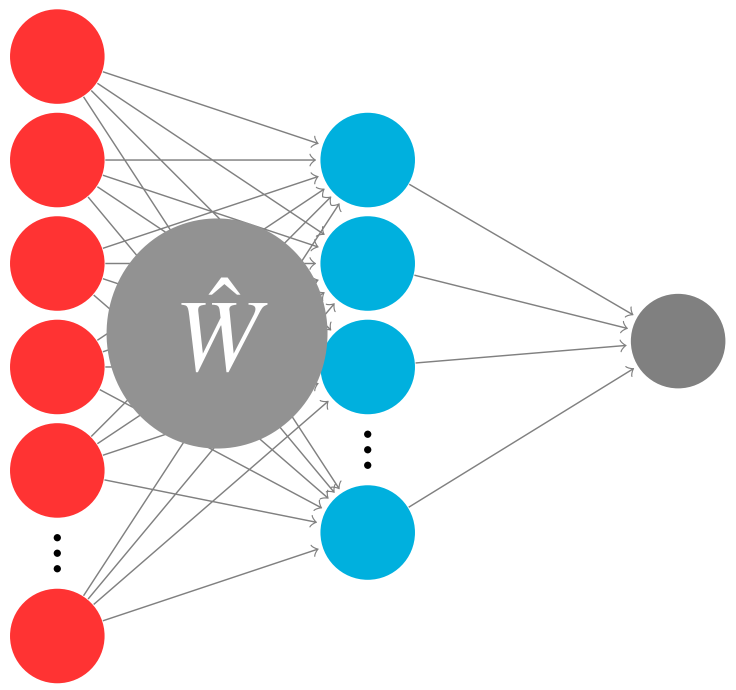

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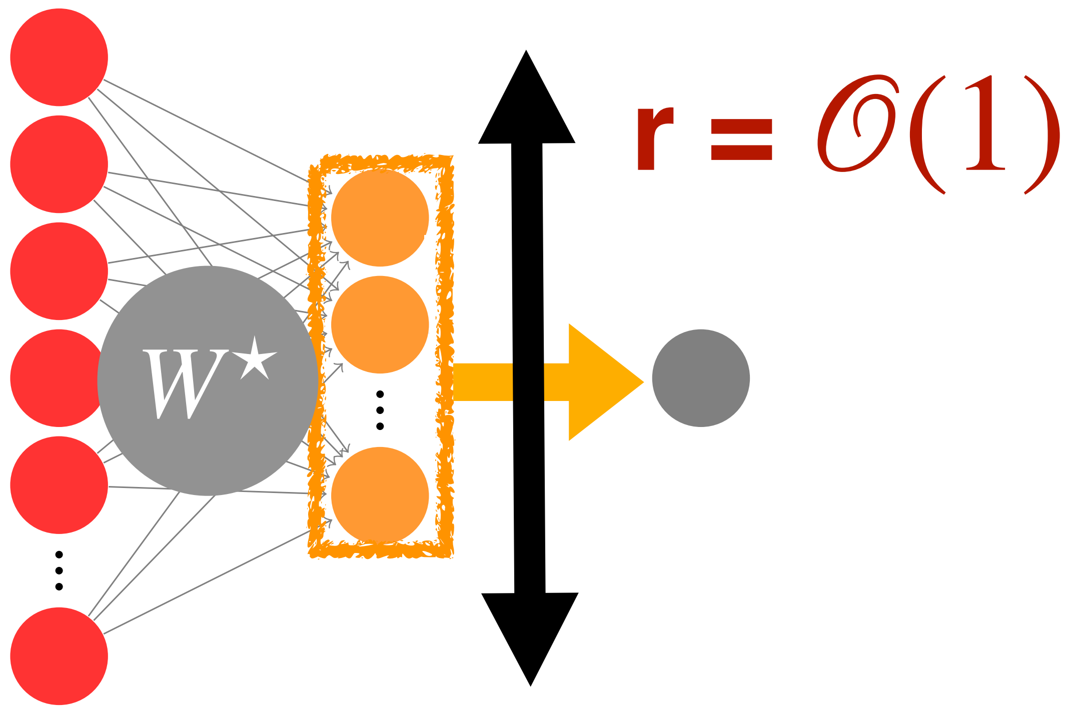

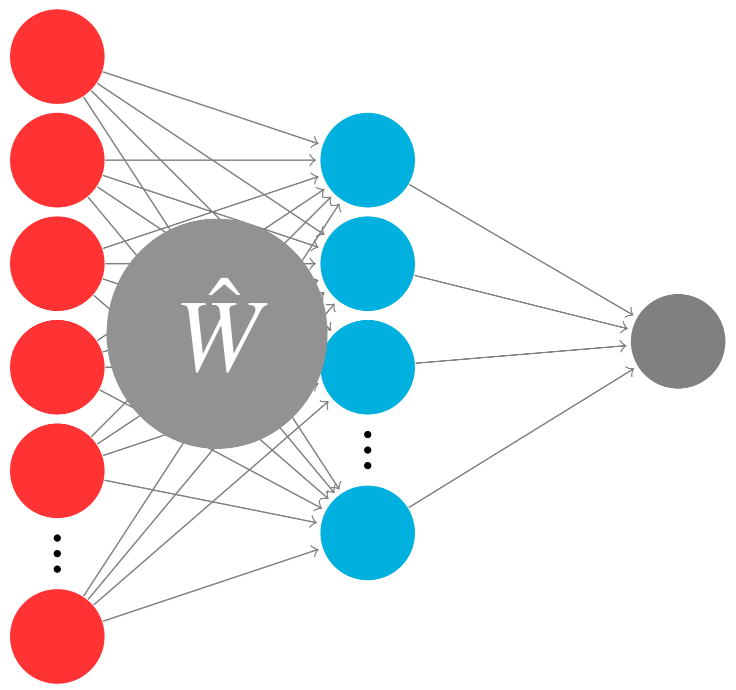

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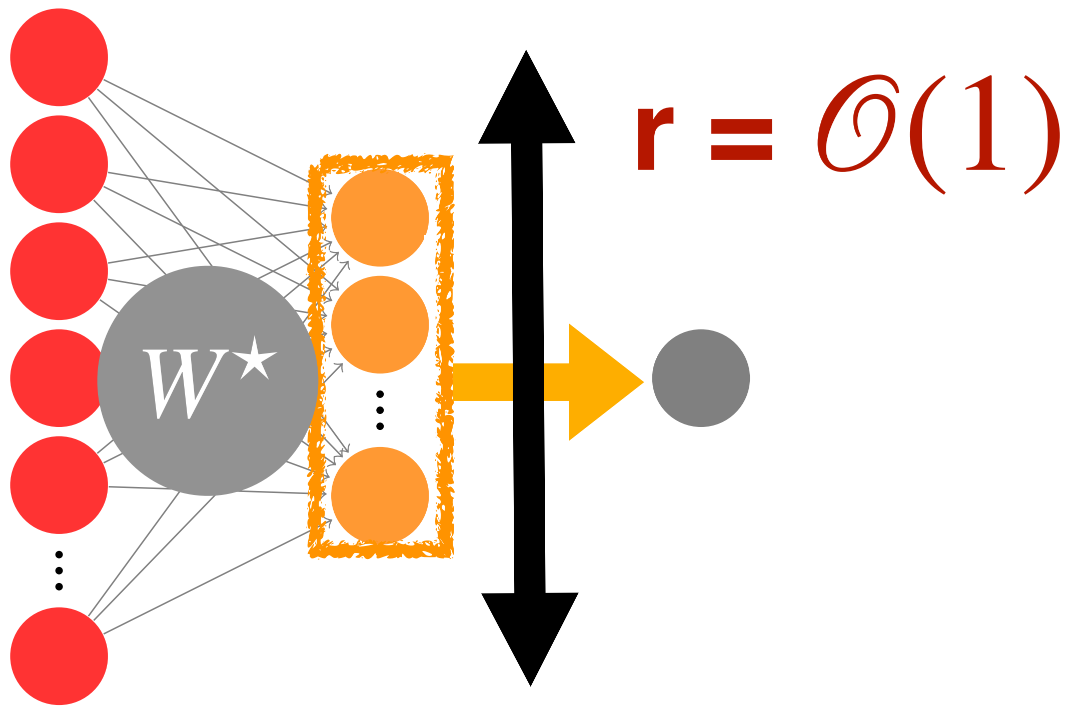

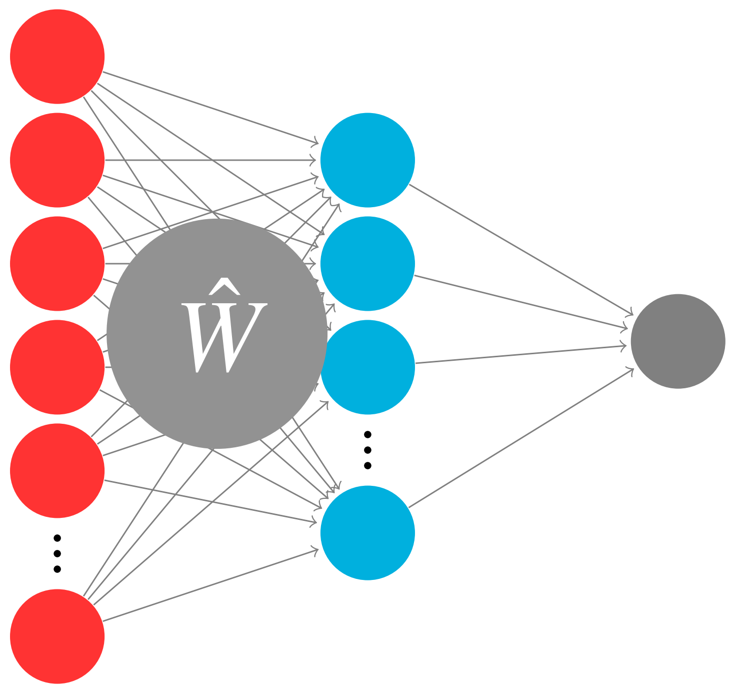

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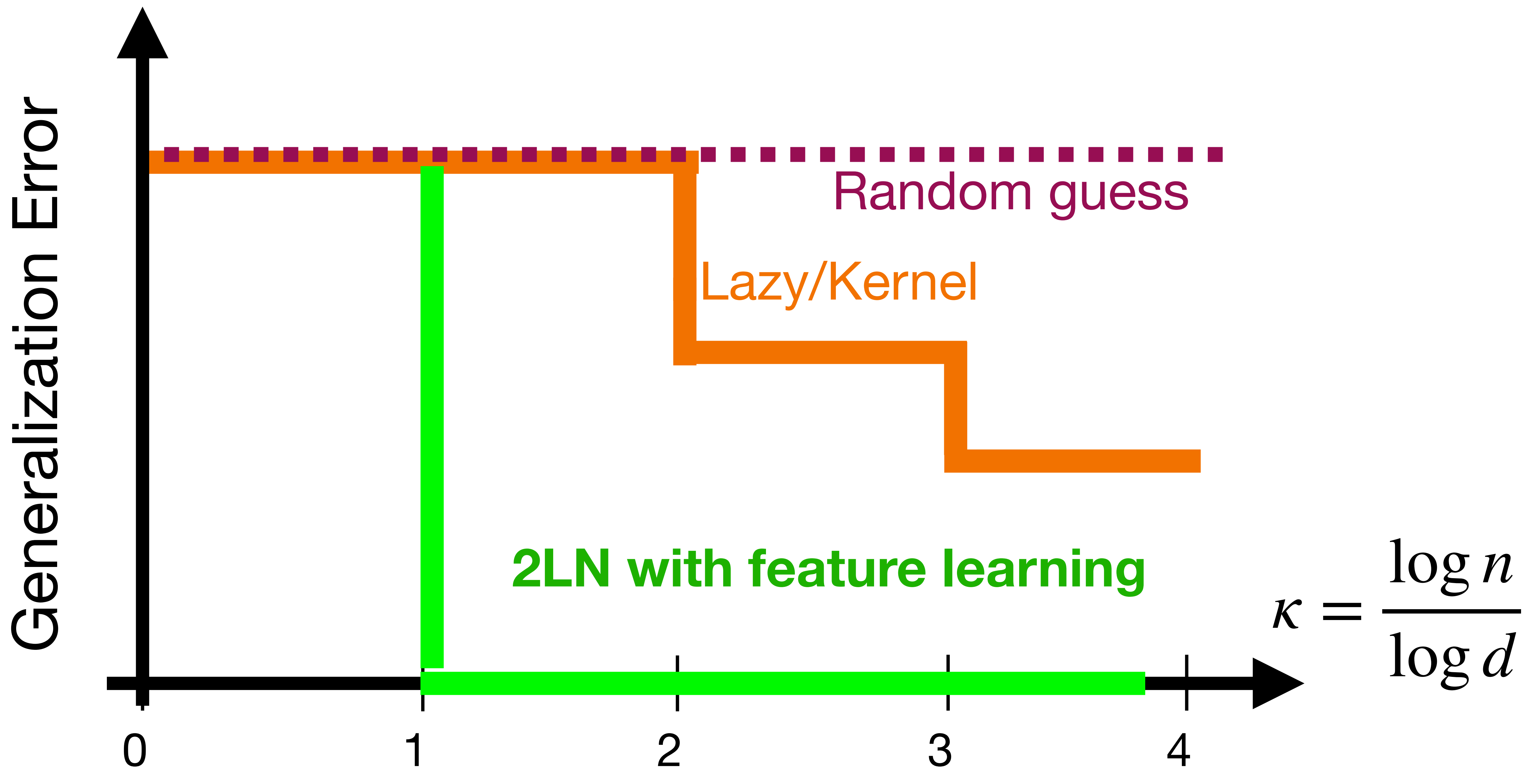
Random feature in (finite) **reduced** space
$d \to d^{\text{eff}} = r = O(1)$

Ben Arous et al. 2021, Abbe et al. 2022,2023, Damian et al. 2022, Dandi et al. 2023, 2024, etc.

Dimensionality reduction

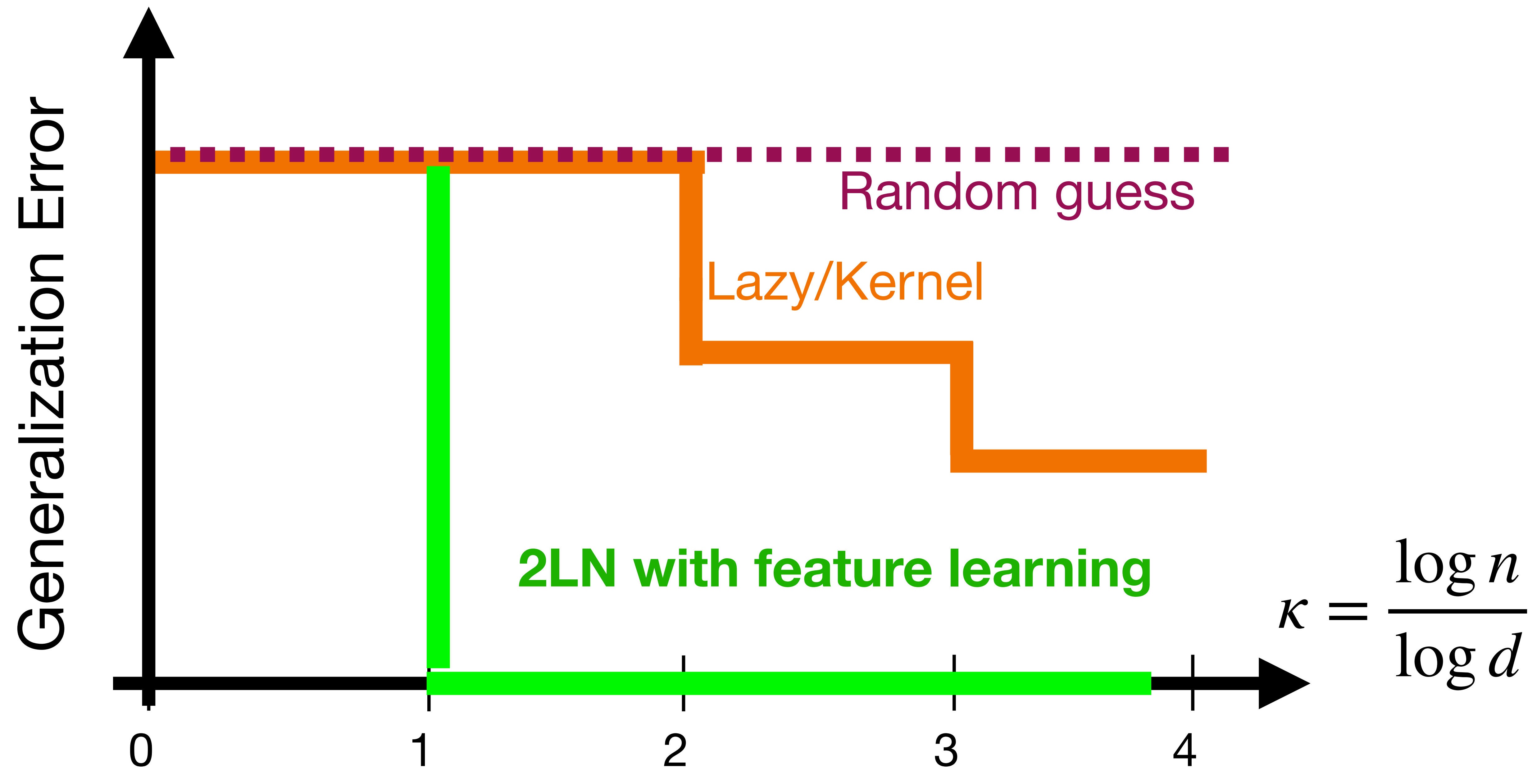


Sample Complexity Reduction

$\mathcal{O}(d^k) \to \tilde{O}(d)$

Dimensionality reduction

searching in a $\mathcal{O}(d^k)$ dimensional subspace of polynomials



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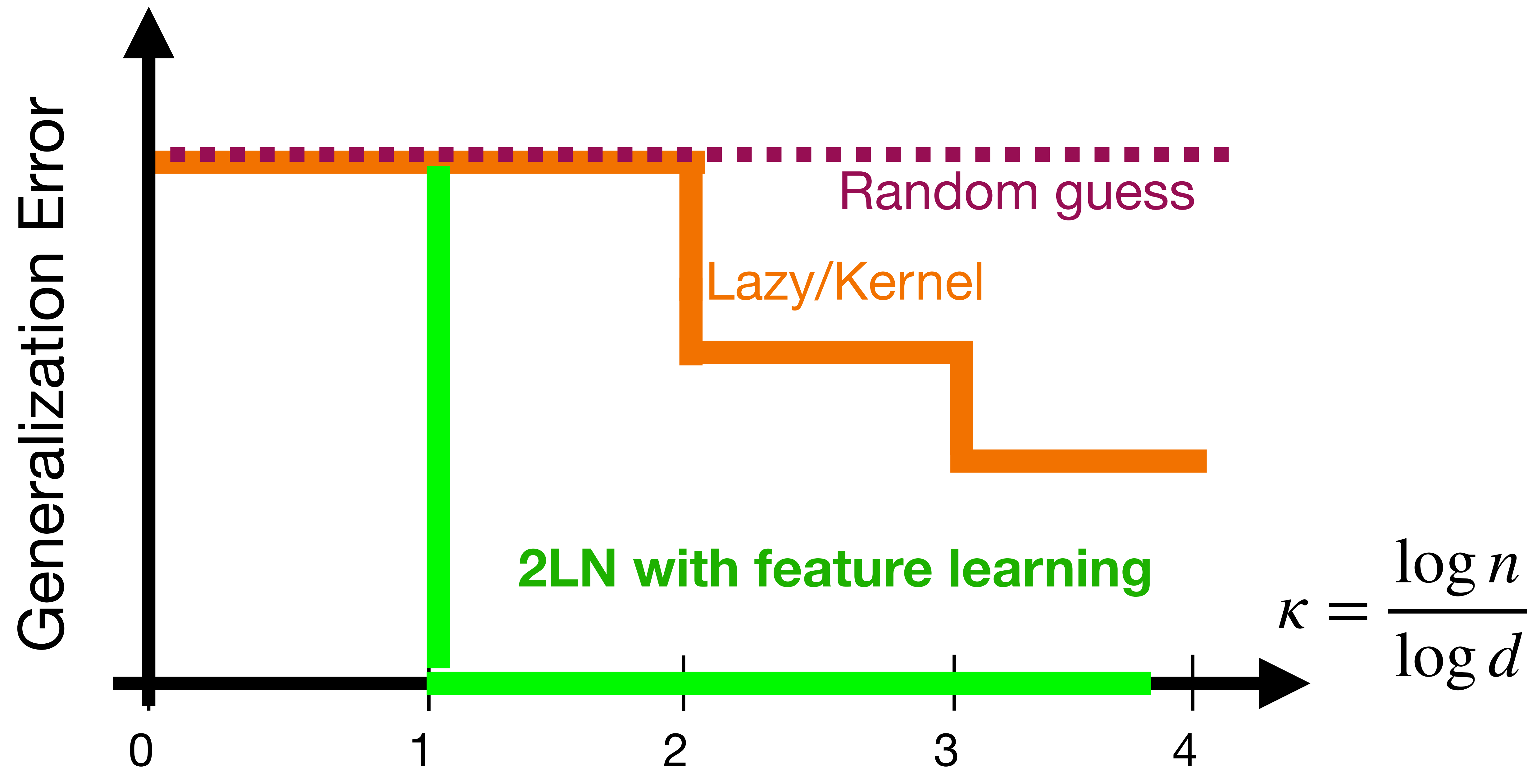

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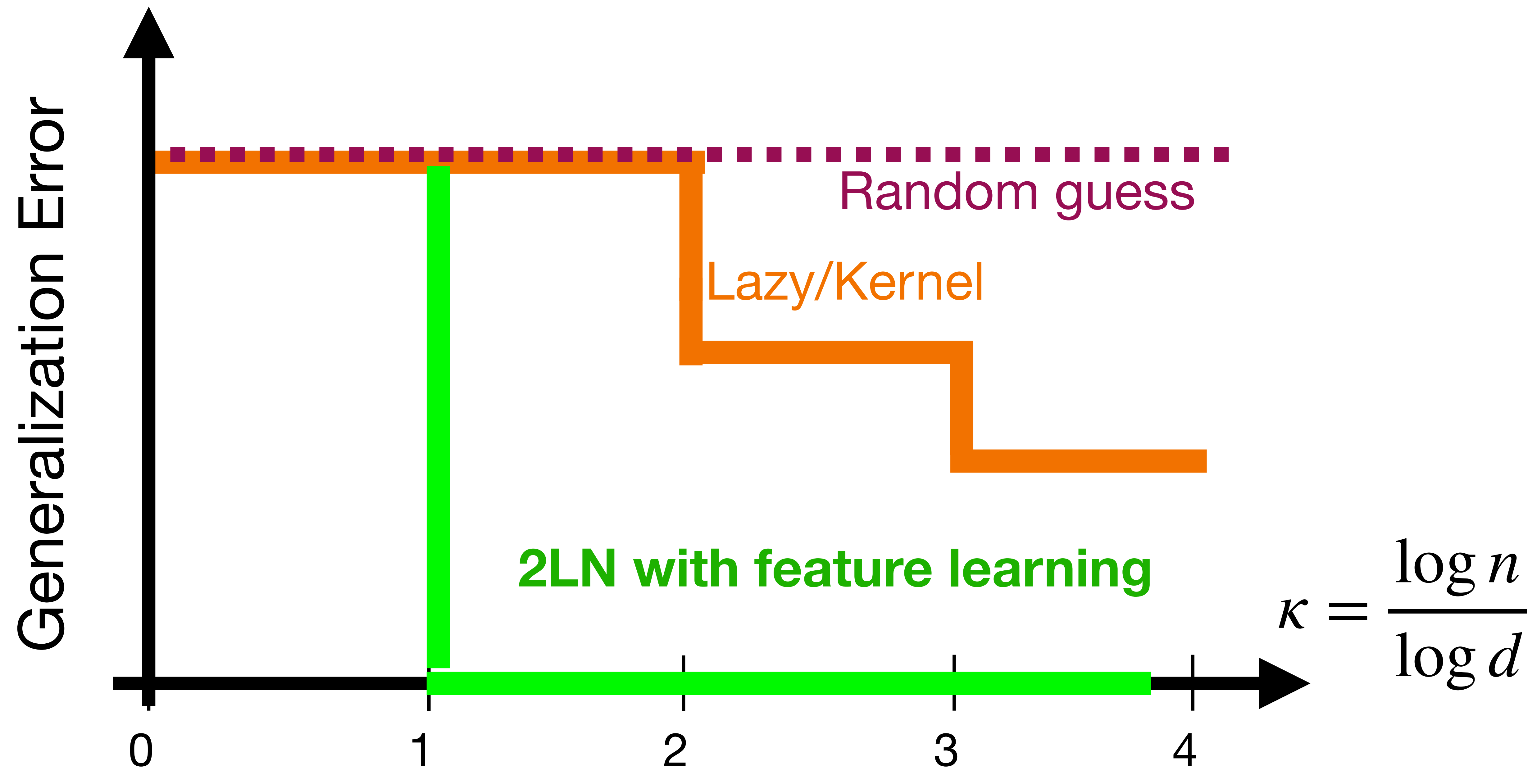

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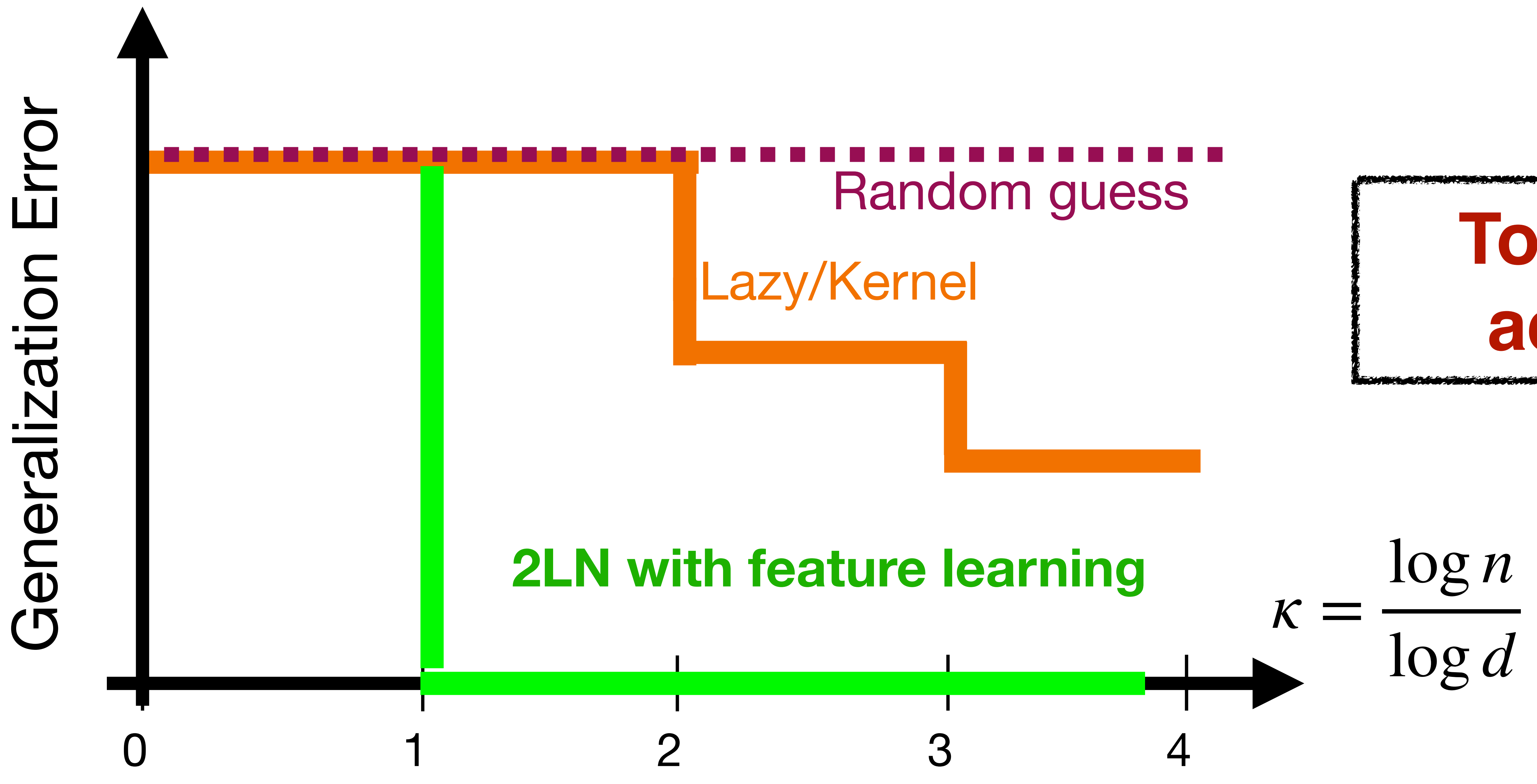


Too easy…no further advantage of depth

Sample Complexity Reduction

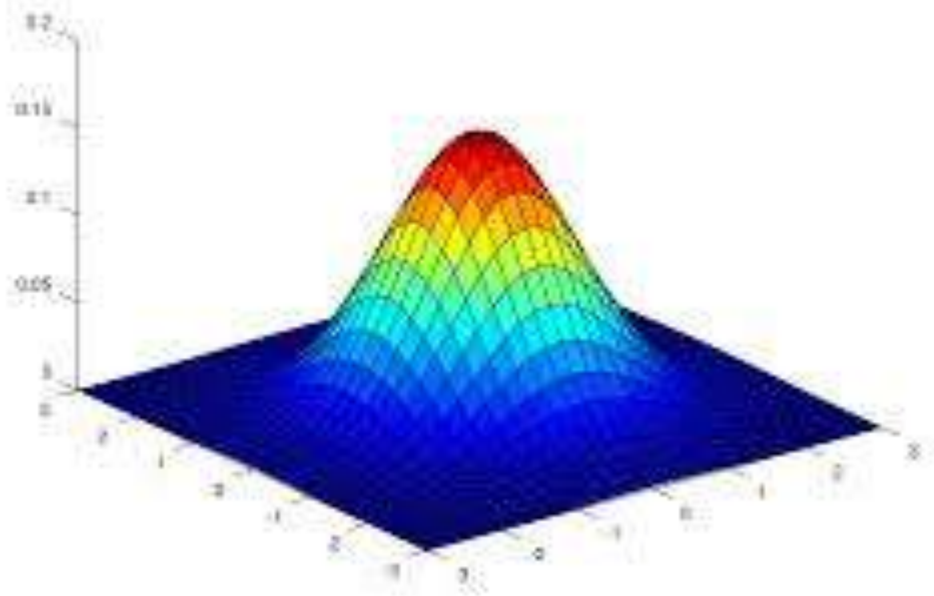
$\mathcal{O}(d^k) \to \tilde{O}(d)$

Can we generalize this picture to arbitrary depth?

A SIGHT beyond Two Layers (Single-Index Gaussian Hierarchical Targets)

$$f^\star(\mathbf{x}) = g^\star\left(\frac{\mathbf{a}^{\star\top} P_k\left(W^\star \mathbf{x}\right)}{\sqrt{d^\varepsilon}}\right), \mathbf{x} \in \mathbb{R}^d$$

P_k: polynomial of degree k
$g^\star$: non-linearity



high-dimensional Gaussian
$\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{I_d})$

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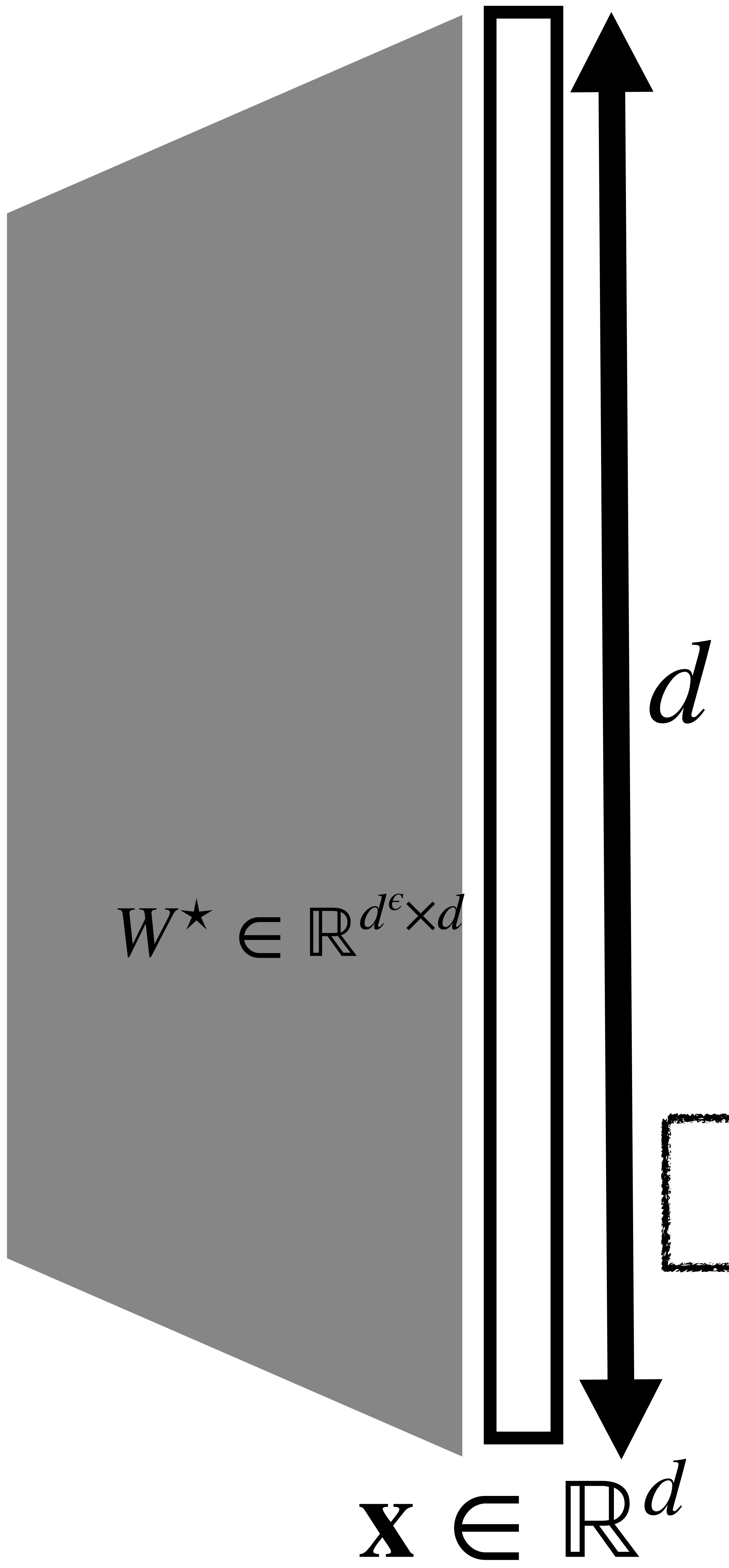

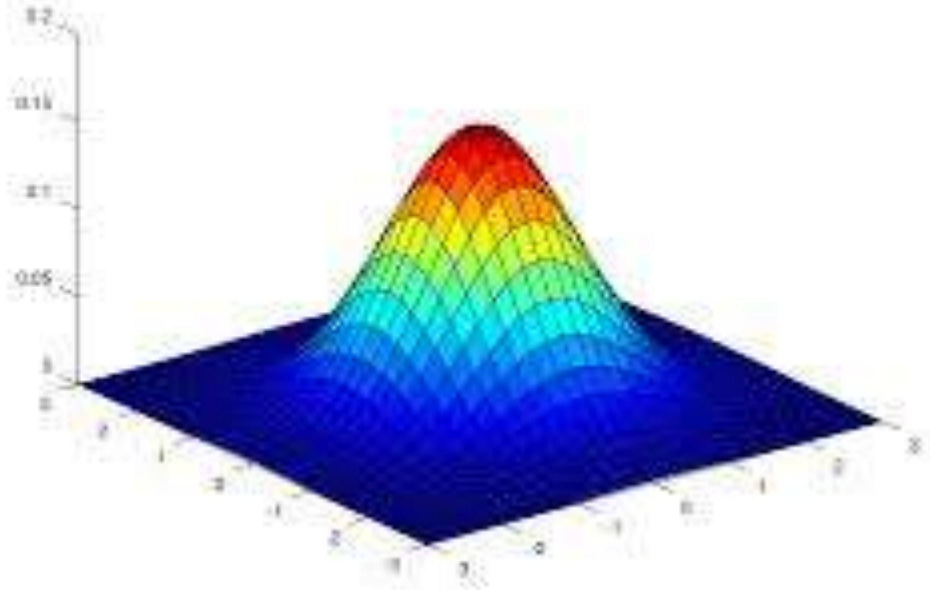

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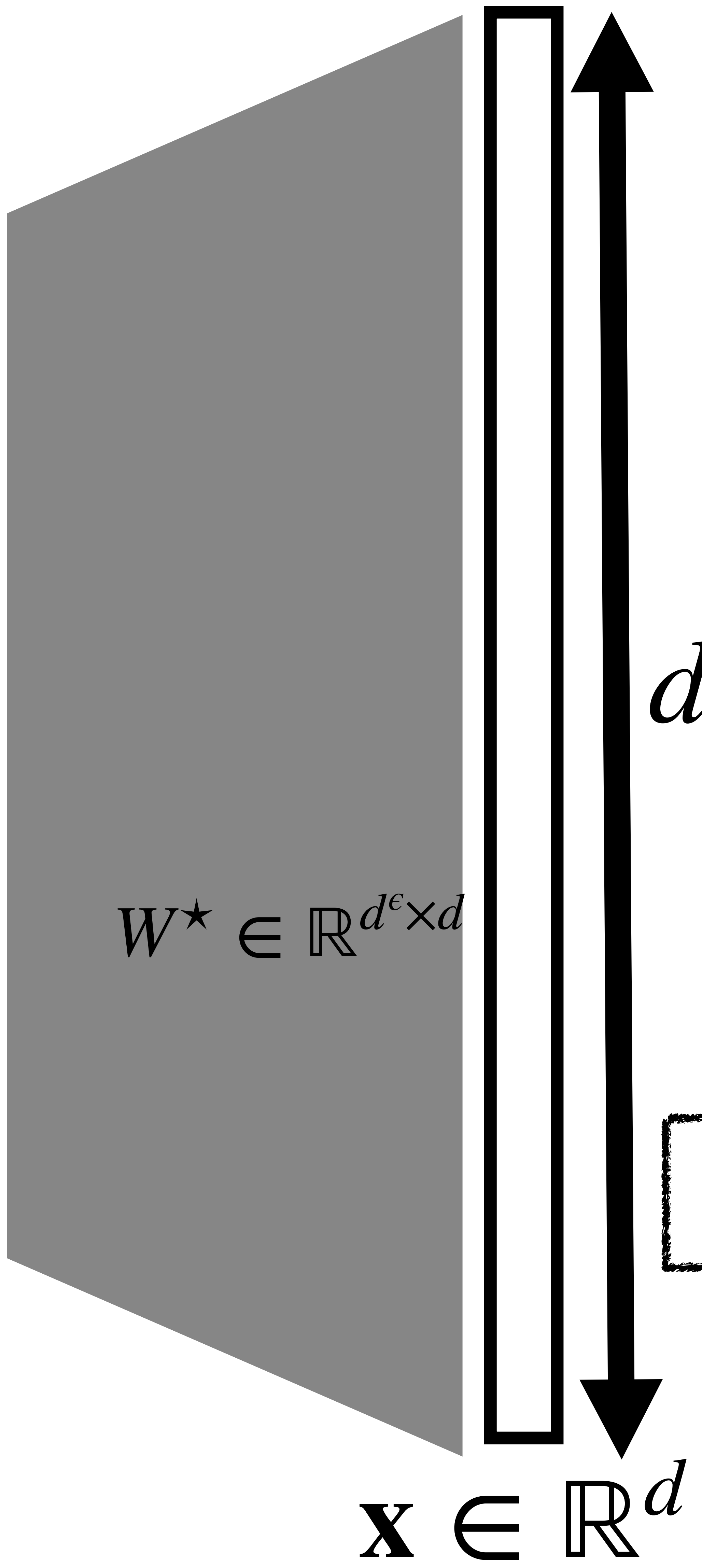

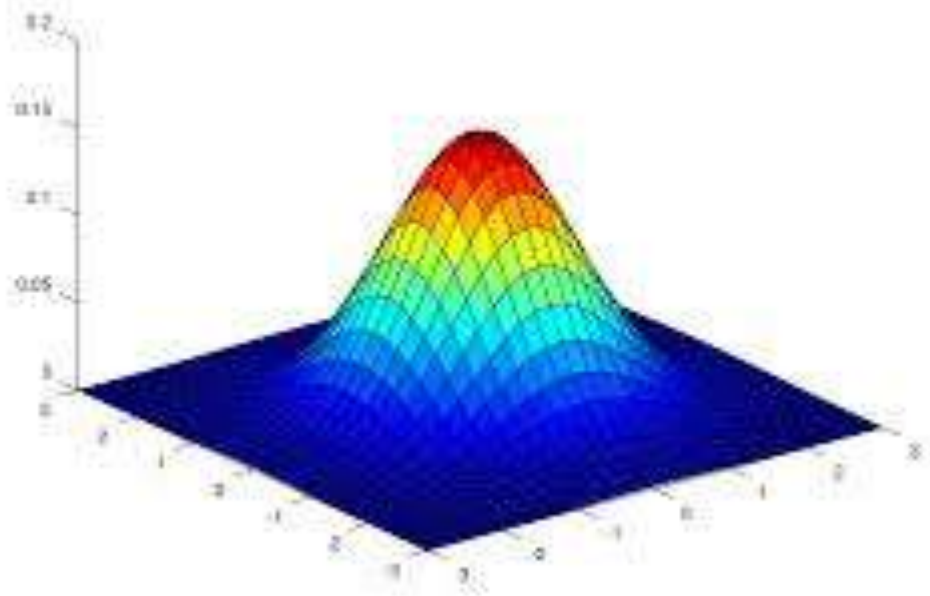

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Subspace dimension d^ϵ
for $0 < \epsilon < 1$

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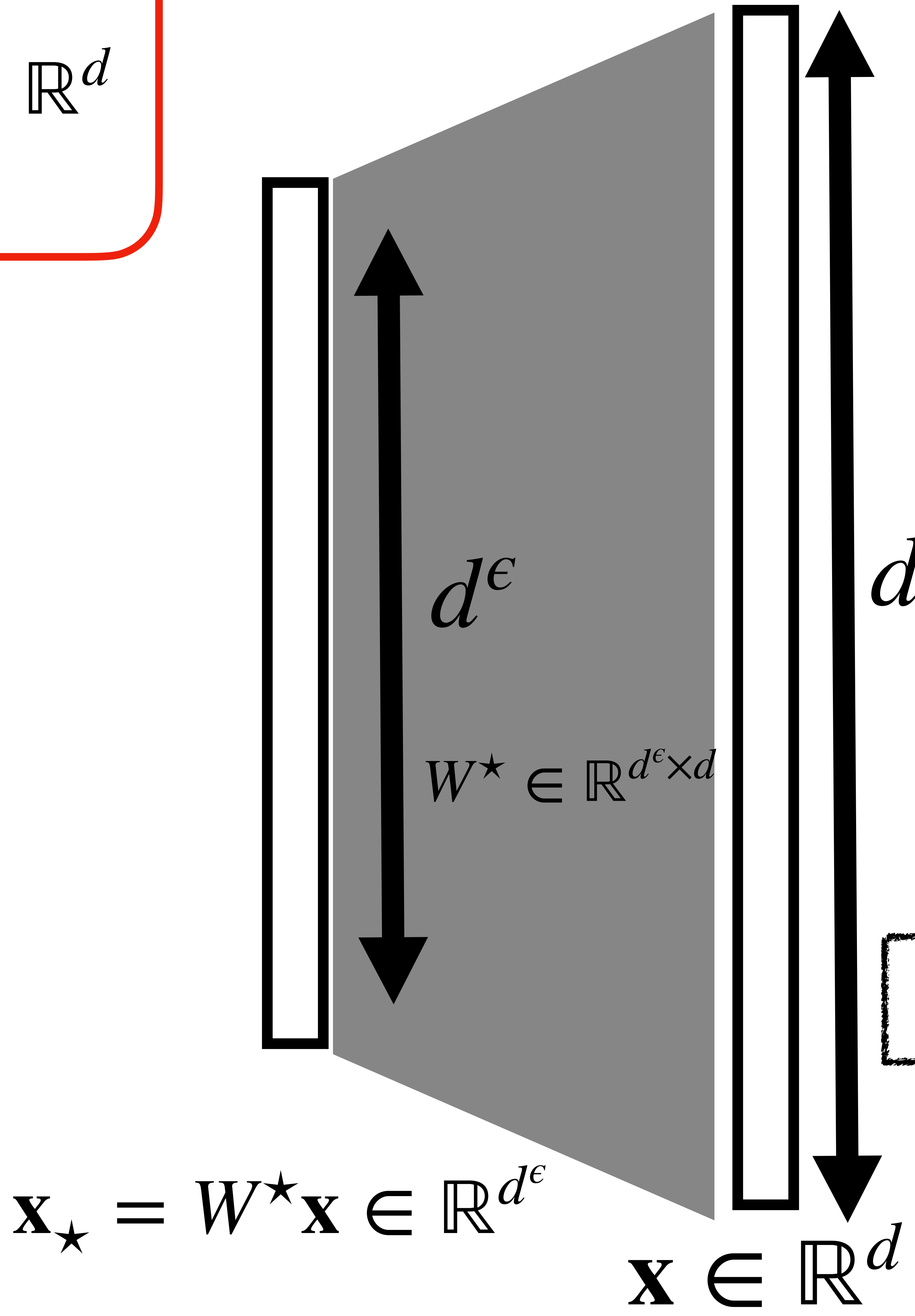

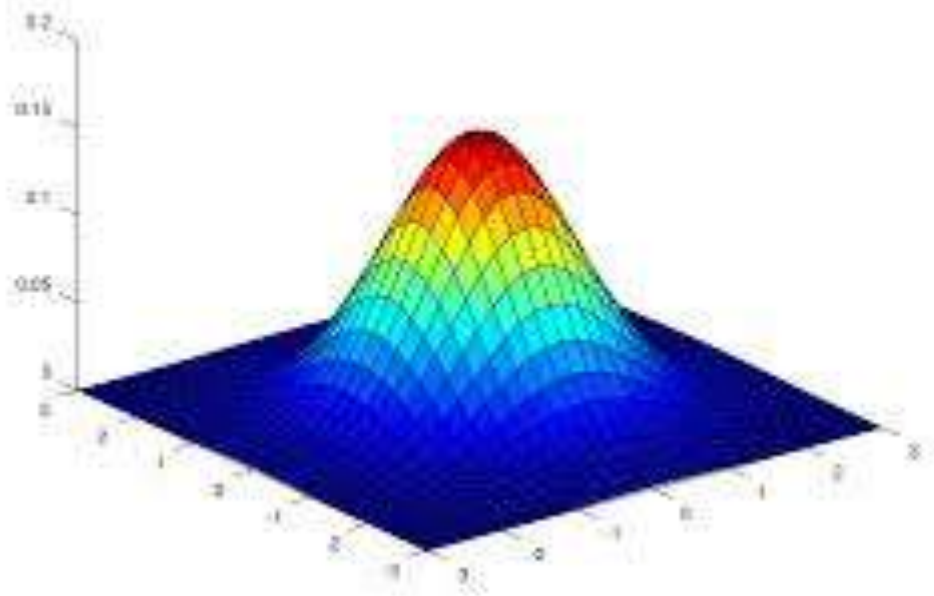

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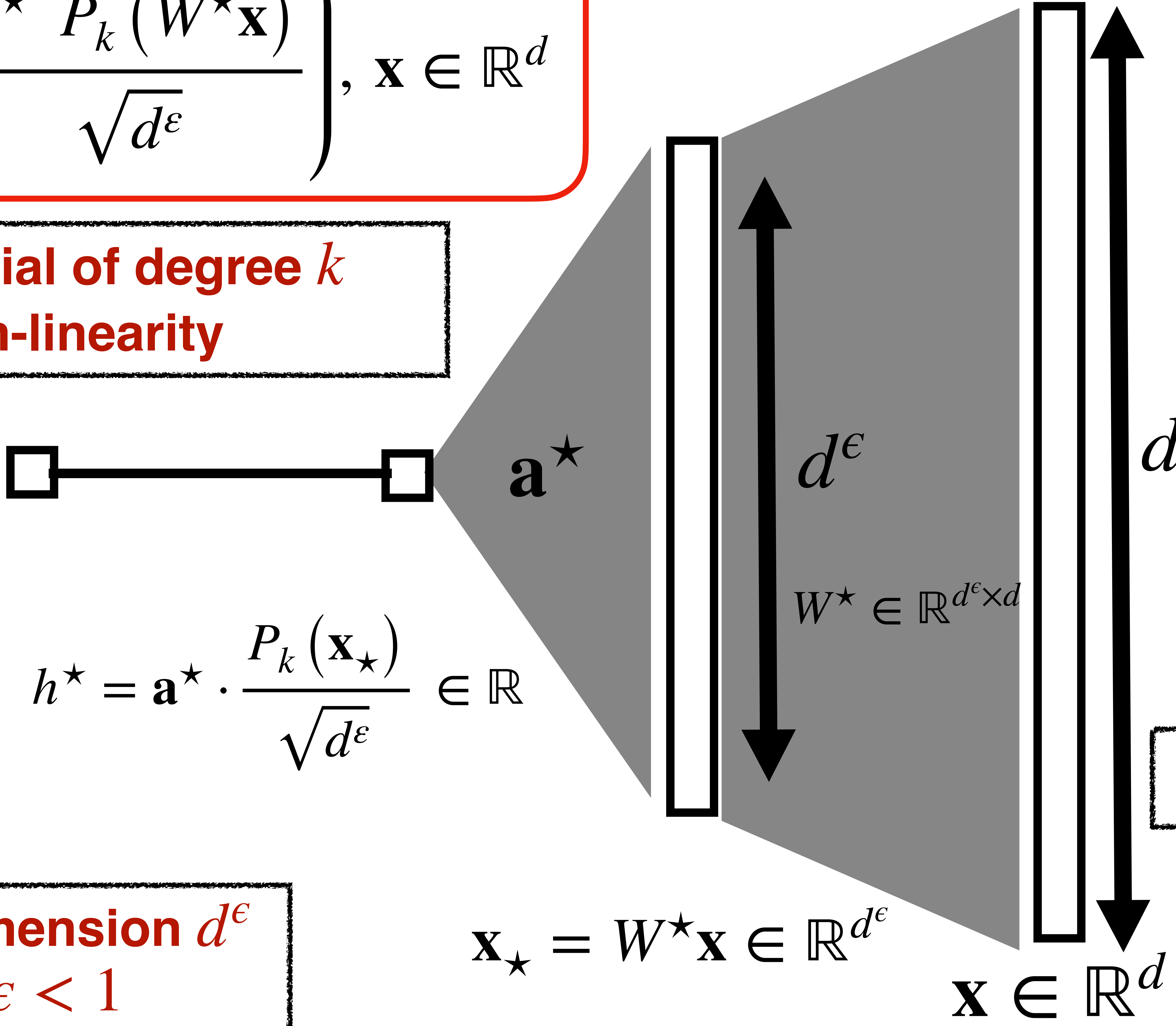
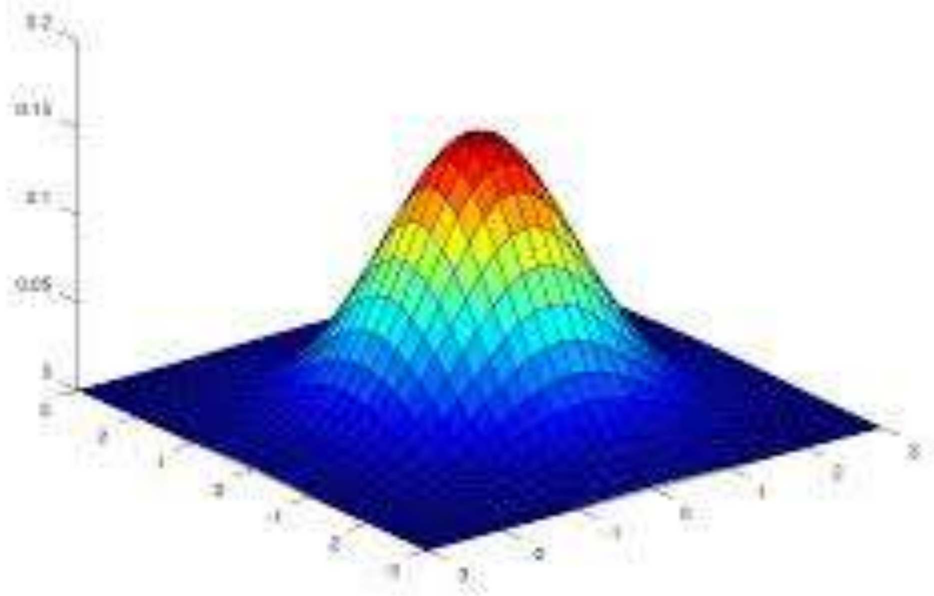
$f^\star(\mathbf{x}) = g^\star(h^\star)$



$$h^\star = \mathbf{a}^\star \cdot \frac{P_k\left(\mathbf{x}_\star\right)}{\sqrt{d^\varepsilon}} \in \mathbb{R}$$



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Subspace dimension d^ϵ
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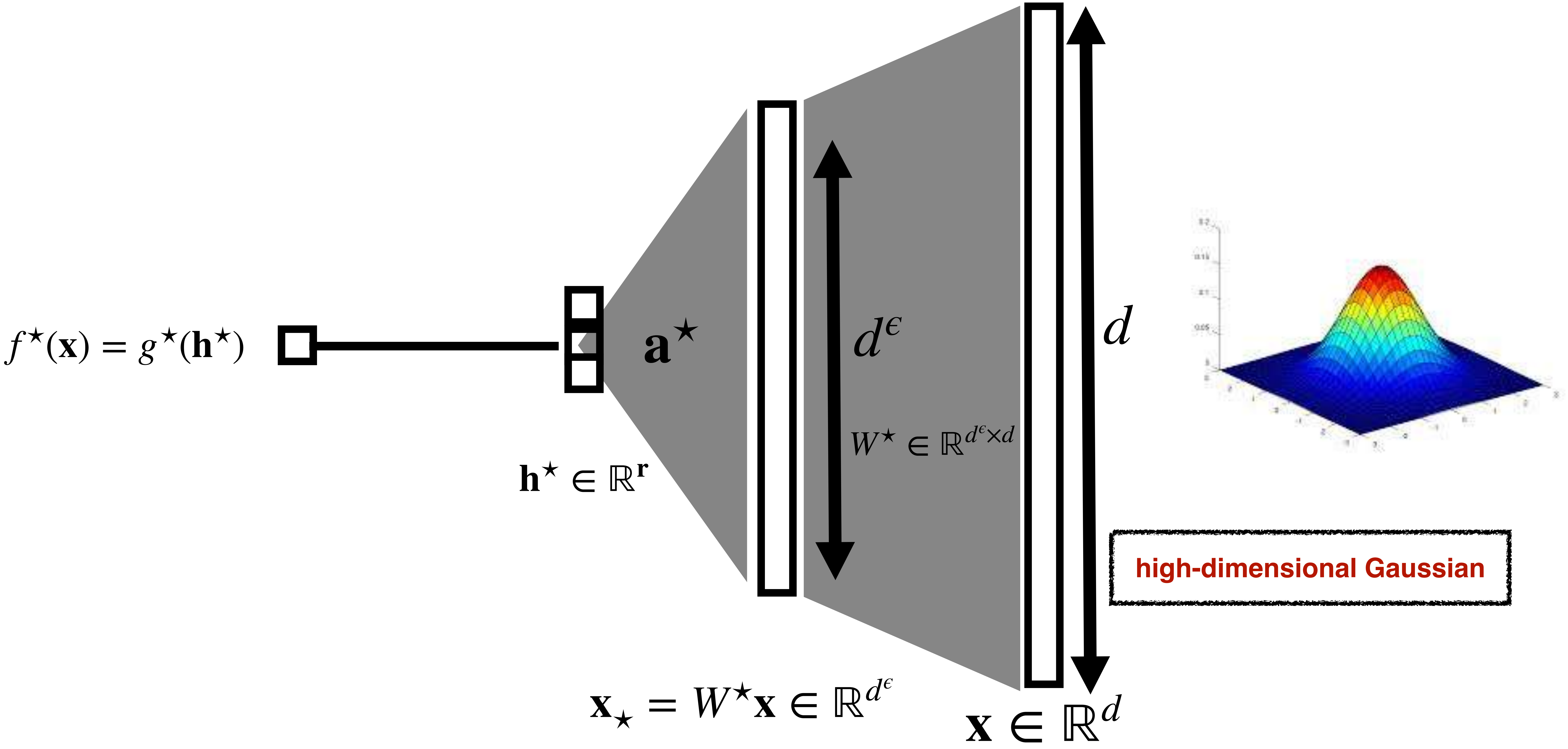
MIGHT (Multi-Index Gaussian Hierarchical Targets)

$$f^\star(\mathbf{x}) = g^\star\left(\mathbf{h}_1^\star(\mathbf{x}), \ldots, \mathbf{h}_r^\star(\mathbf{x})\right), \; \mathbf{x} \in \mathbb{R}^d$$

$$h_m^\star(\mathbf{x}) = \frac{\mathbf{1}}{\sqrt{\mathbf{d}^\varepsilon}} \mathbf{a}_\mathbf{m}^{\star\top} \mathbf{P}_{\mathbf{k,m}}\left(\mathbf{W}_\mathbf{m}^\star \mathbf{x}\right), \mathbf{m} = \mathbf{1}, \ldots \mathbf{r}$$



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2-layer NN can't learn non-linear features

1 level of dimension-reduction

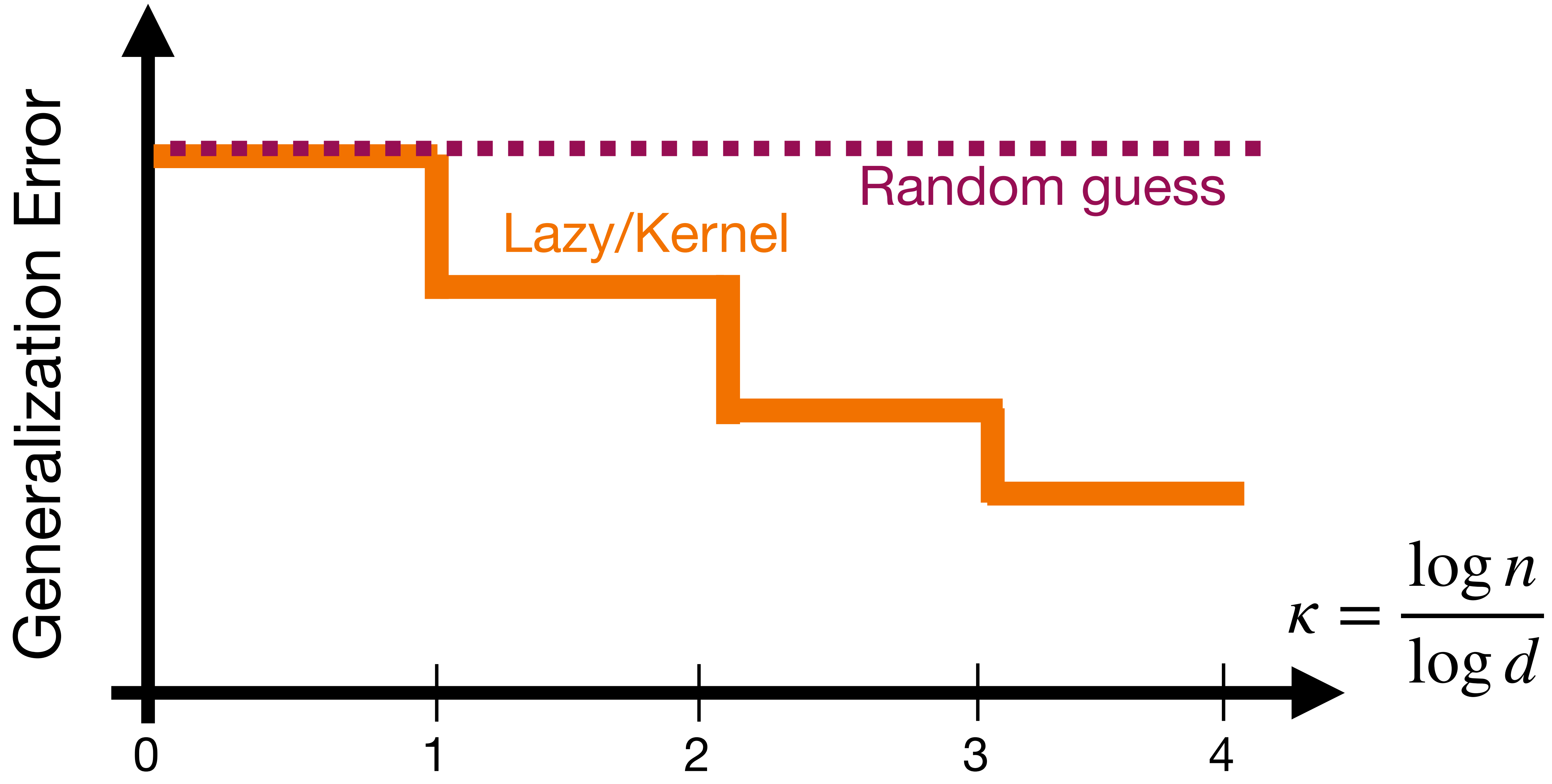
No adaptivity of first layer required

III

Learning SIGHT with Two-layer NNs

Lazy learning (Random Features/NTK)

$$f^\star(\mathbf{x}) = g^\star\left(\frac{\mathbf{a}^{\star\top} P_k\left(W^\star \mathbf{x}\right)}{\sqrt{d^\varepsilon}}\right), \mathbf{x} \in \mathbb{R}^d$$



Learning with a two-layer net

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parameter count of $W^\star$

$$\hat{y} = \hat{W}_2 \sigma(\hat{W}_1 \mathbf{x})$$

With $n = \mathcal{O}(d^{1+\epsilon})$ **data,** $\hat{W}_1$ **can learn the features with GD :** $\hat{W}_1 \approx Z_1 W^\star + Z_2$

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Adaptive Learning with a two-layer net

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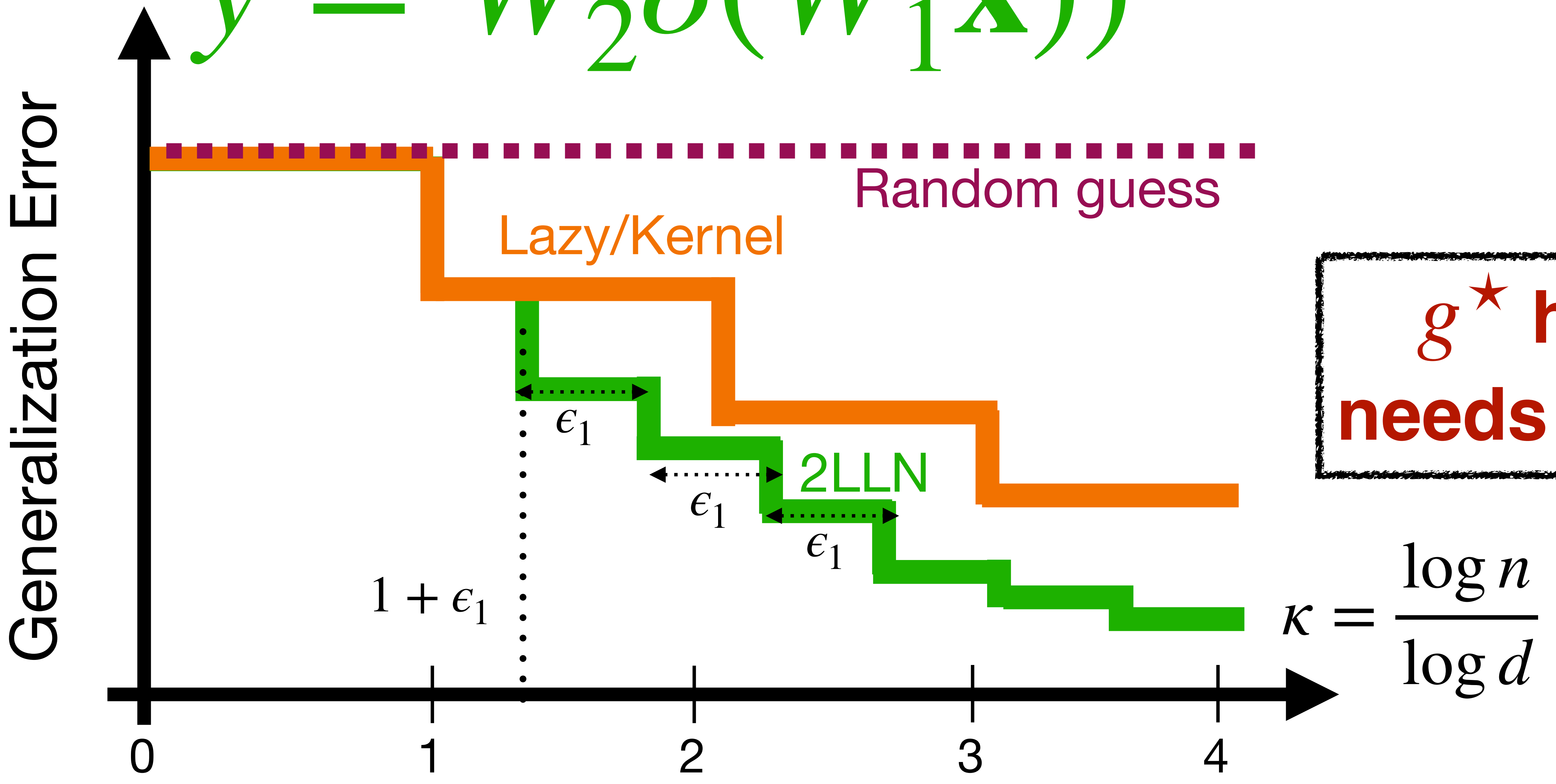
$$\hat{y} = \hat{W}_2 \sigma(\hat{W}_1 \mathbf{x}))$$

Only 1 level of dimension reduction



$g^\star$ has degree m $\Longrightarrow$ needs sample, width $d^{km\epsilon_1}$

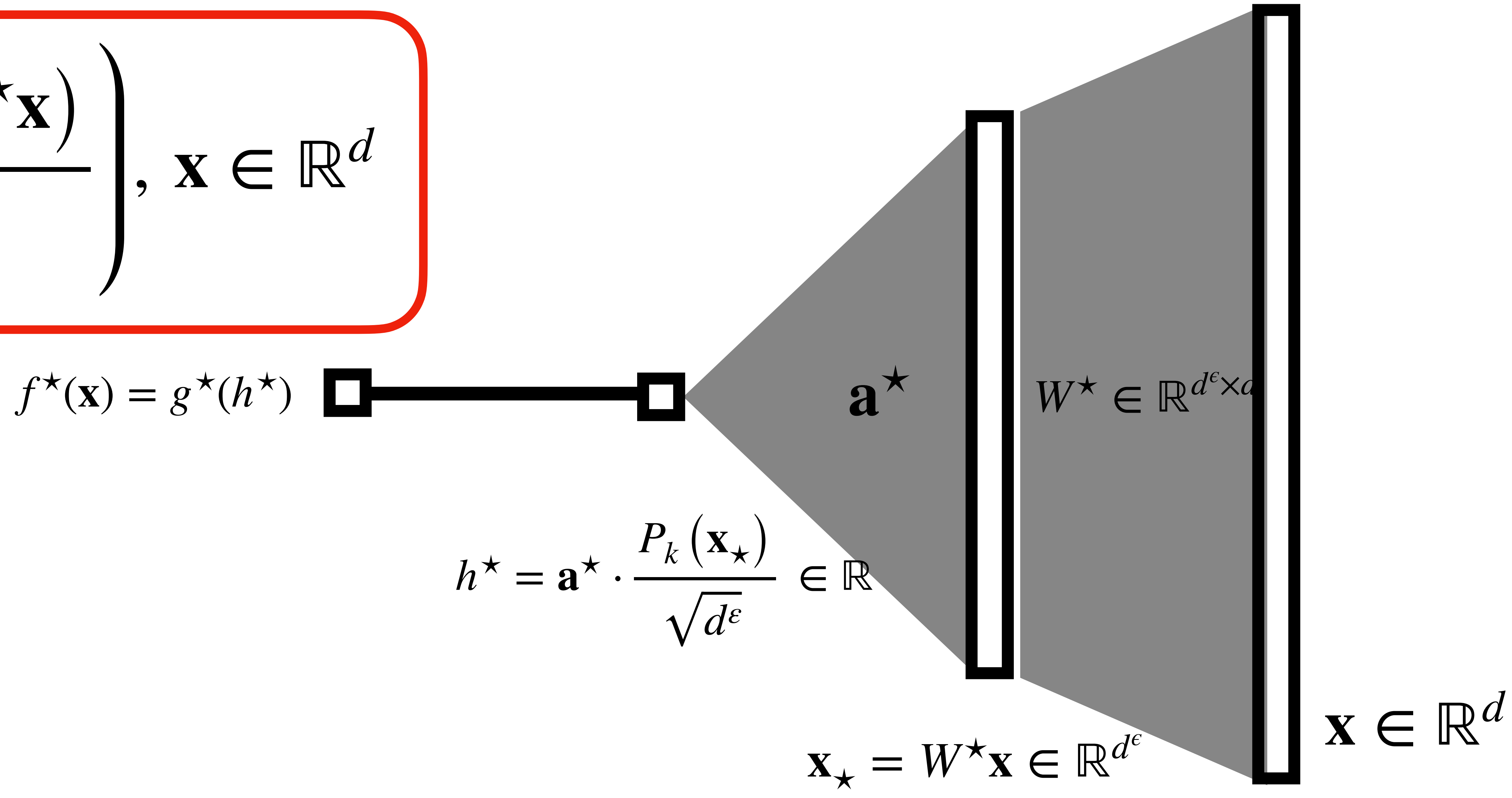
random feature/lazy learning but in reduced dimension $\mathbb{R}^{d^\epsilon}$

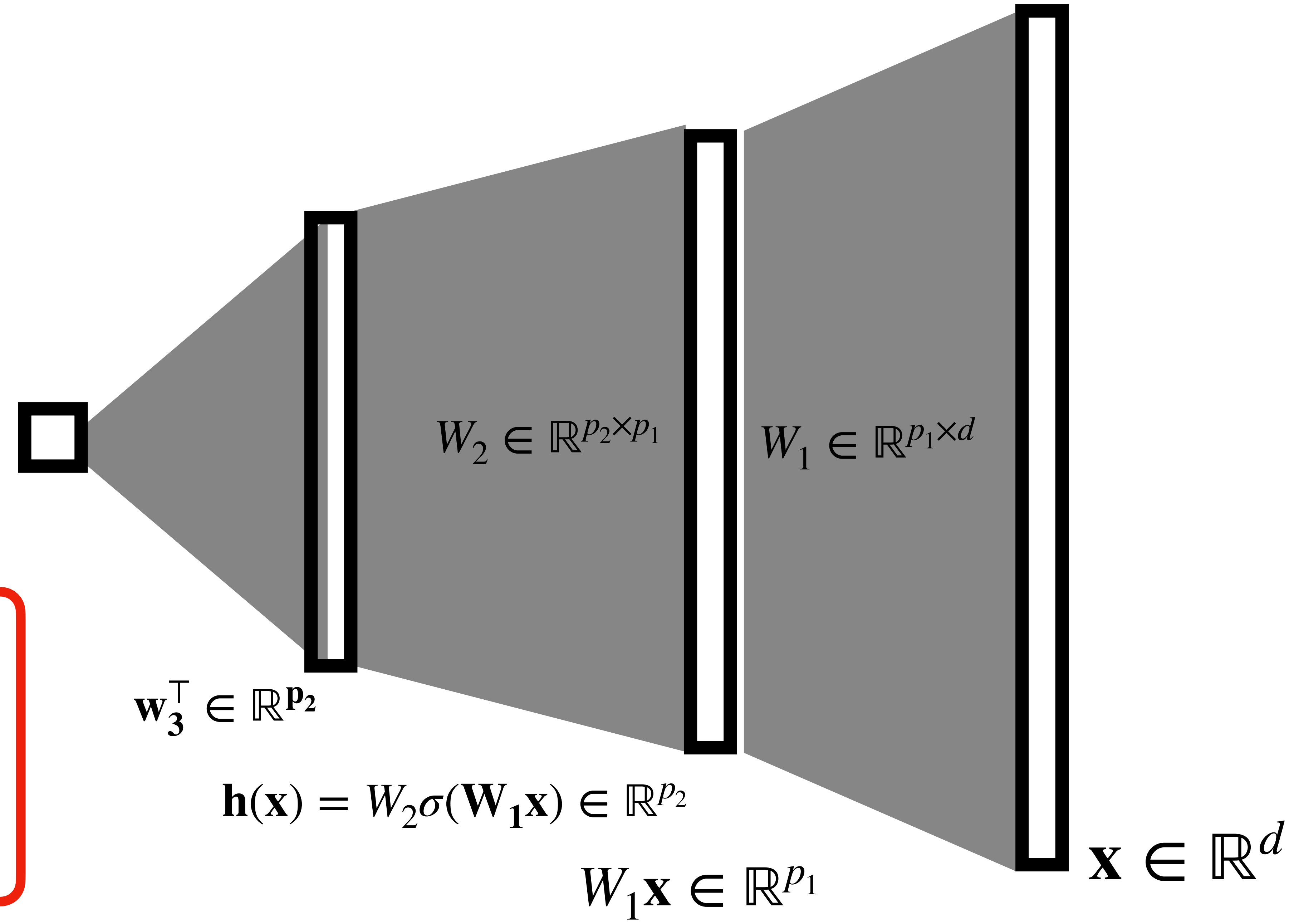
IV

Learning SIGHT with Three-layer NNs

Idealized scenario for layer-wise training in 3-layer NNS

$$f^\star(\mathbf{x}) = g^\star\left(\frac{\mathbf{a}^{\star\top} P_k\left(W^\star \mathbf{x}\right)}{\sqrt{d^{\varepsilon_1}}}\right), \mathbf{x} \in \mathbb{R}^d$$

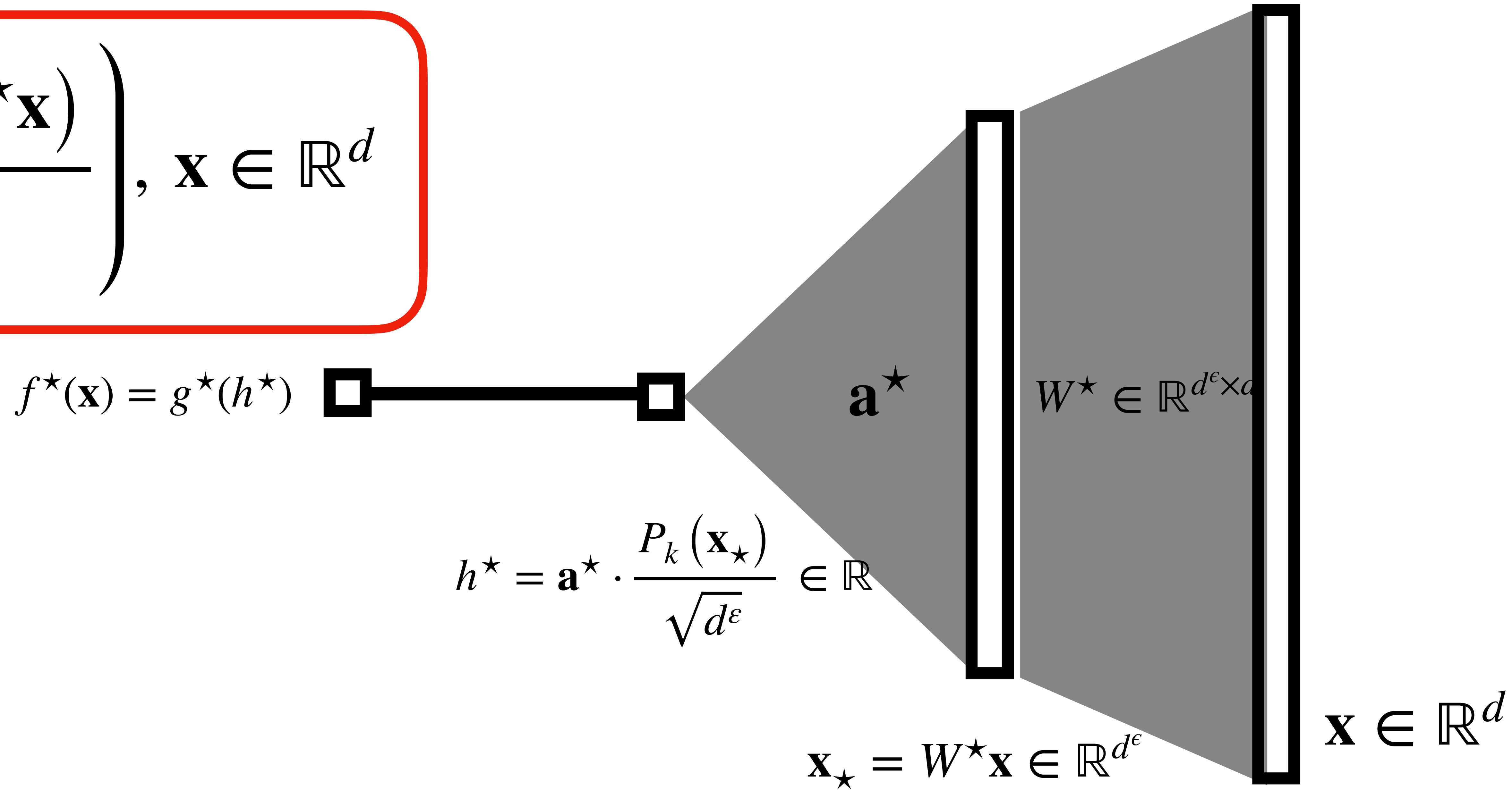


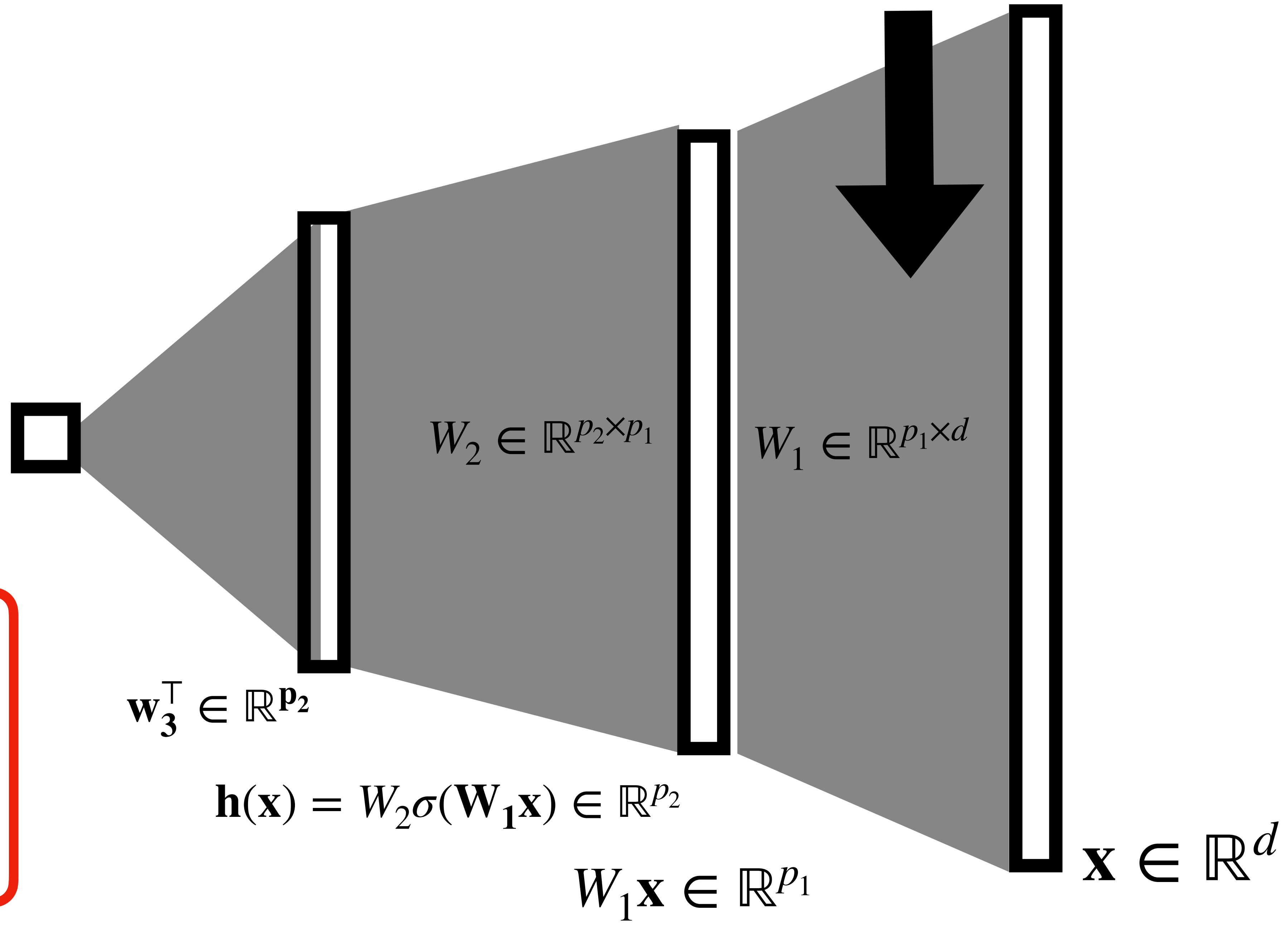


$$\hat{f}(\mathbf{x}) = \mathbf{w_3}^\top \sigma\left(W_2\sigma(W_1\mathbf{x} + b_1) + b_2\right)$$

Idealized scenario for layer-wise training in 3-layer NNS

$$f^\star(\mathbf{x}) = g^\star\left(\frac{\mathbf{a}^{\star\top} P_k\left(W^\star\mathbf{x}\right)}{\sqrt{d^{\varepsilon_1}}}\right),\ \mathbf{x}\in\mathbb{R}^d$$





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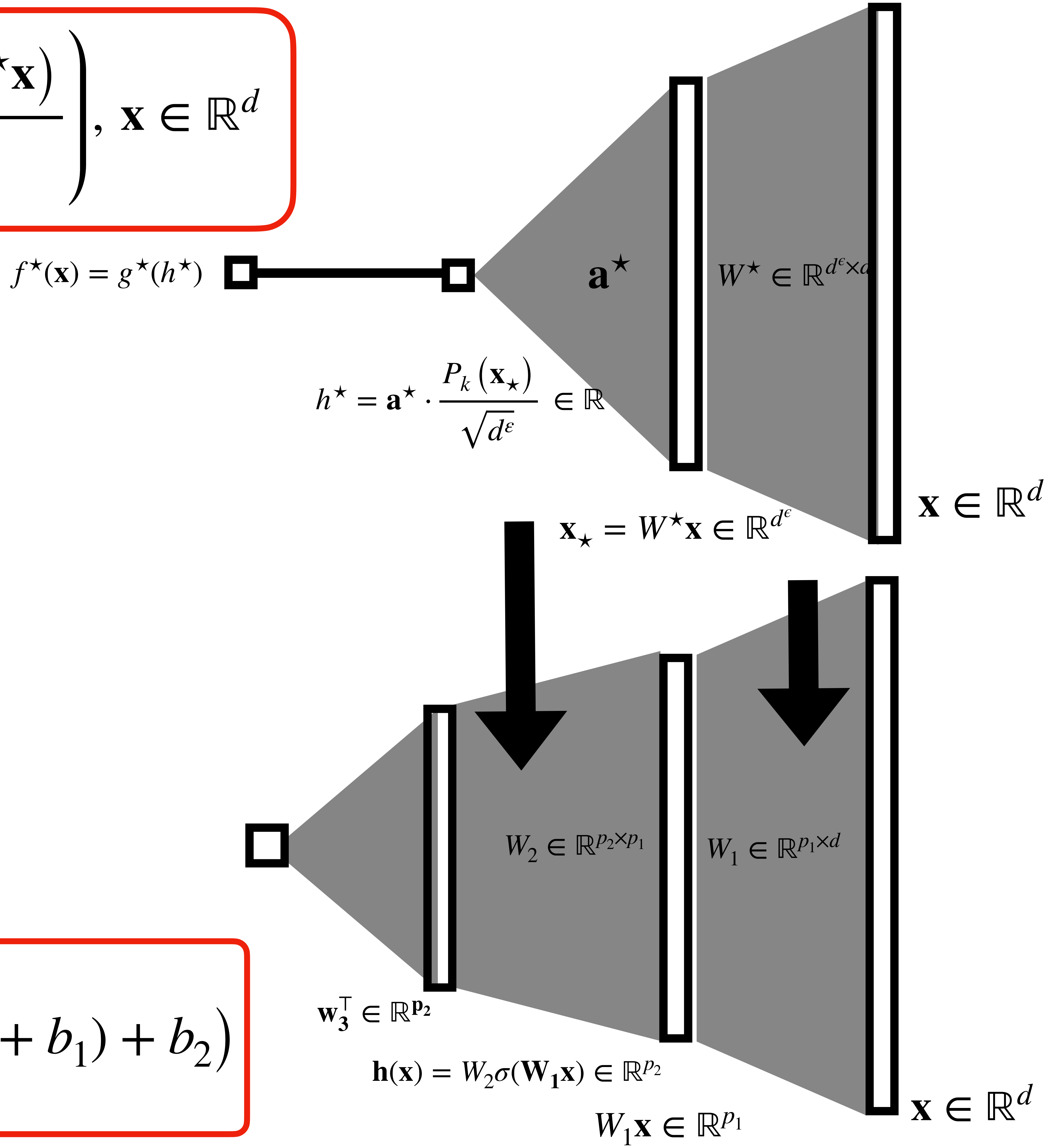

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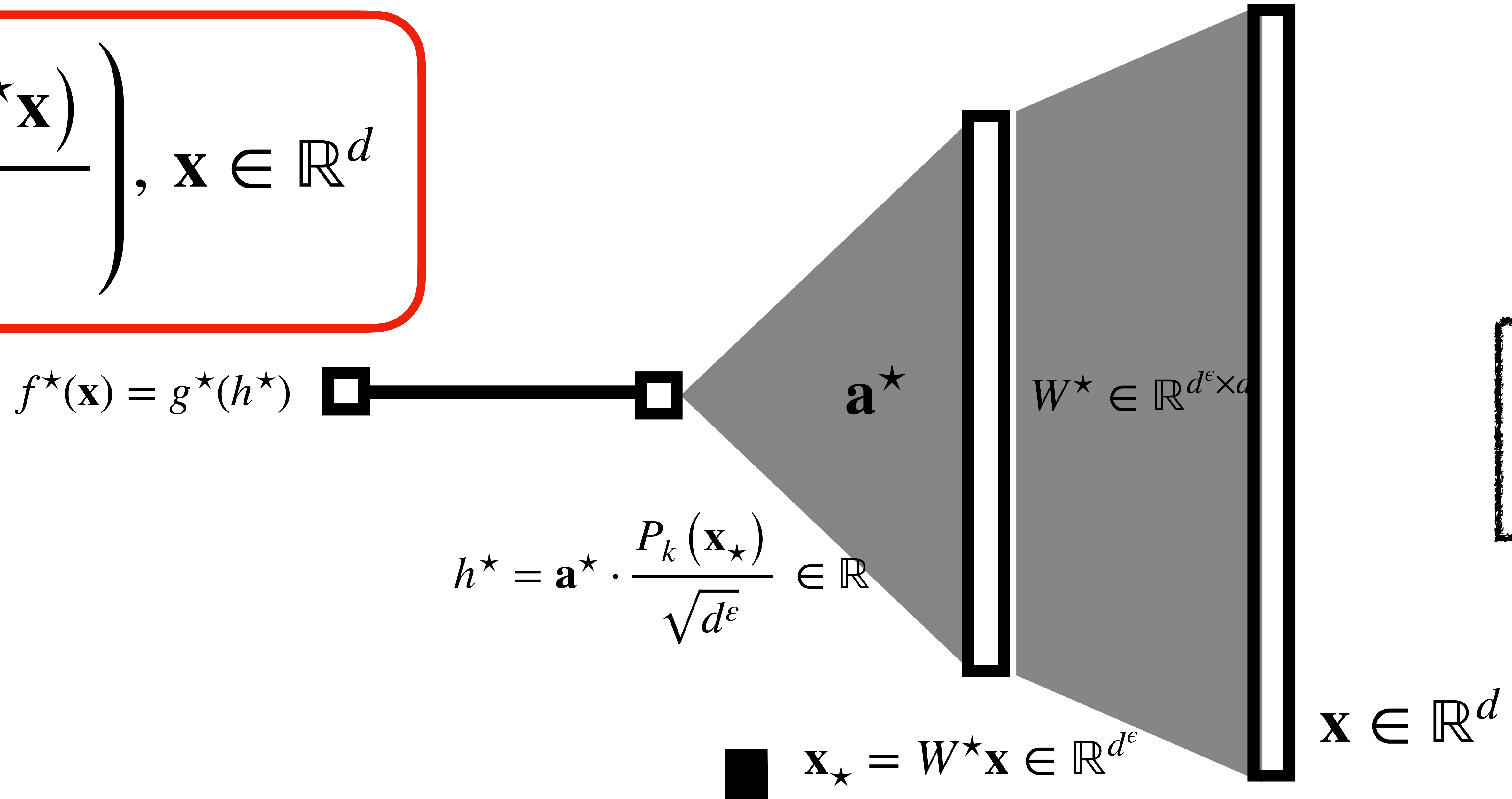
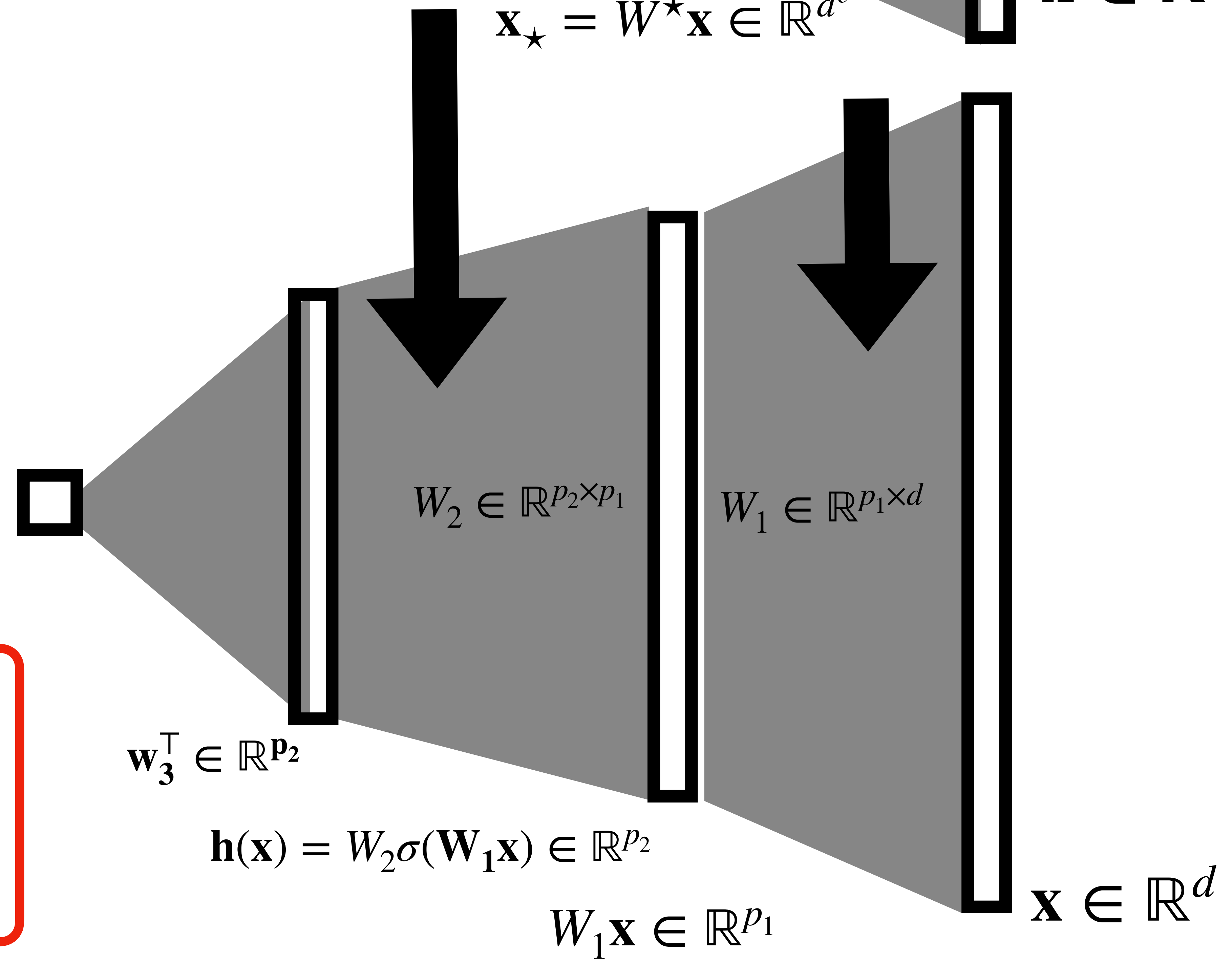


W_2 learns features not weights $a^\star$



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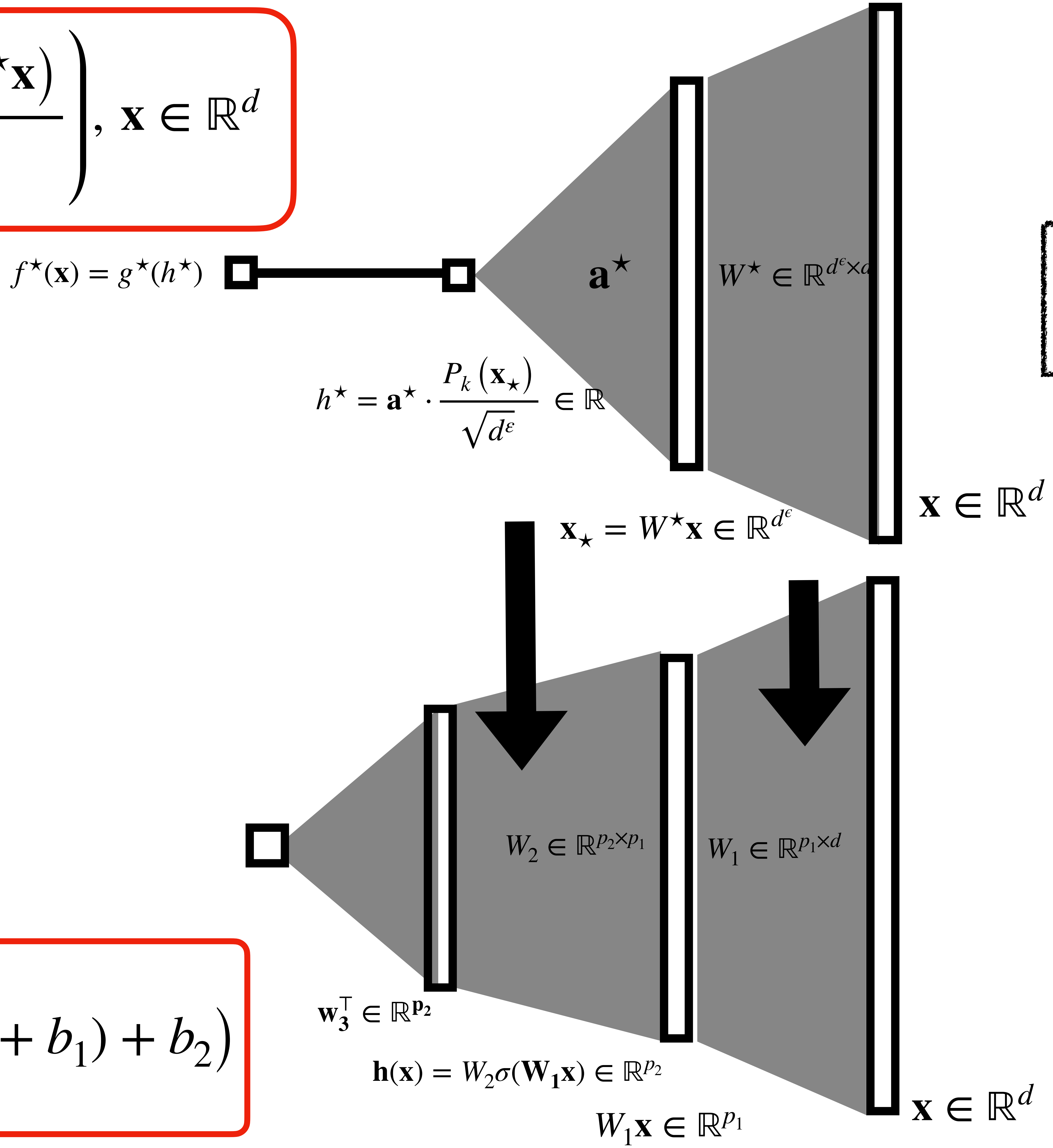

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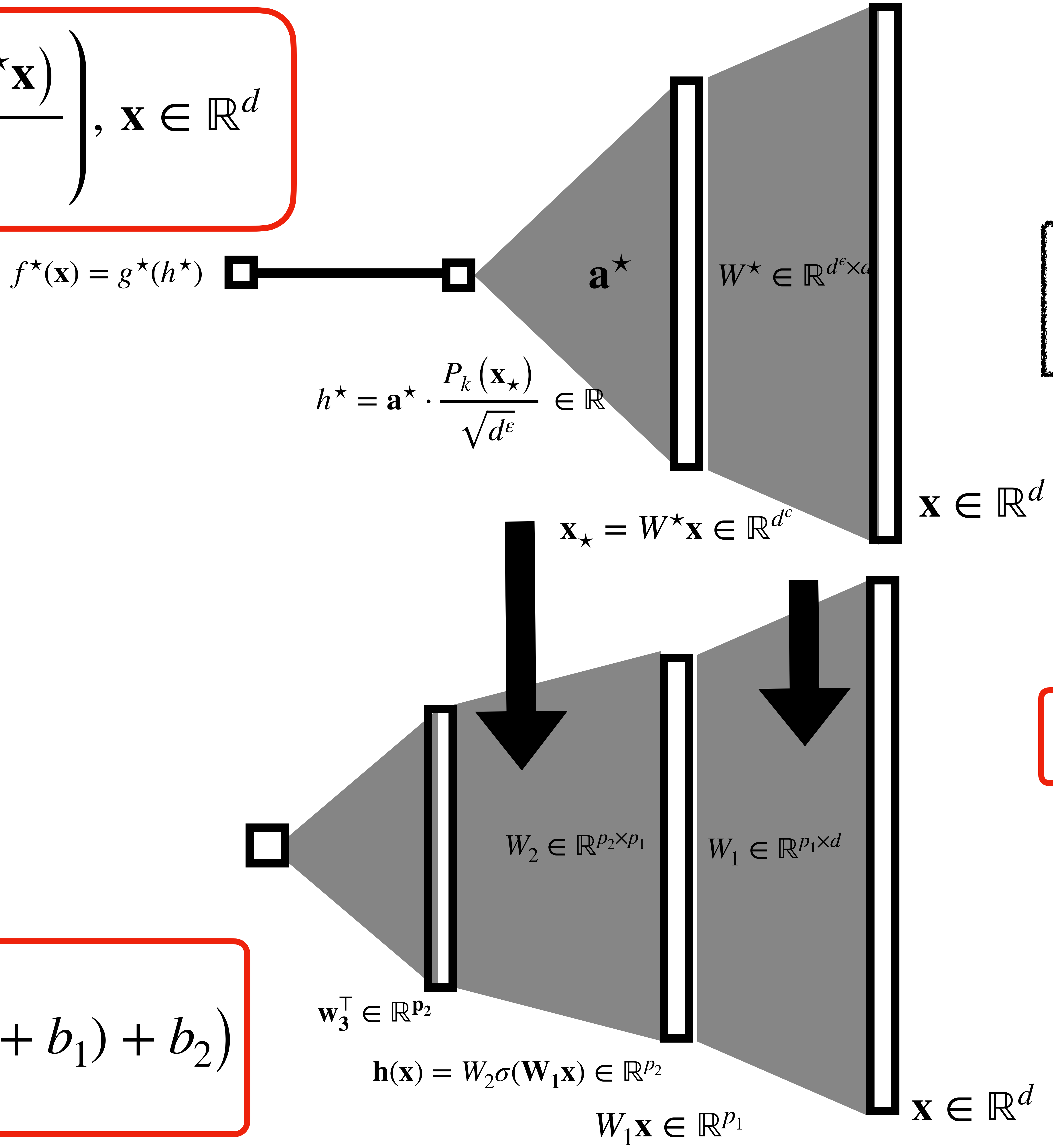

W_2 learns features not weights $a^\star$

$$\mathbf{h}(\mathbf{x}) = W_2\sigma(\mathbf{W}_1\mathbf{x}) \propto h^\star(\mathbf{x})$$

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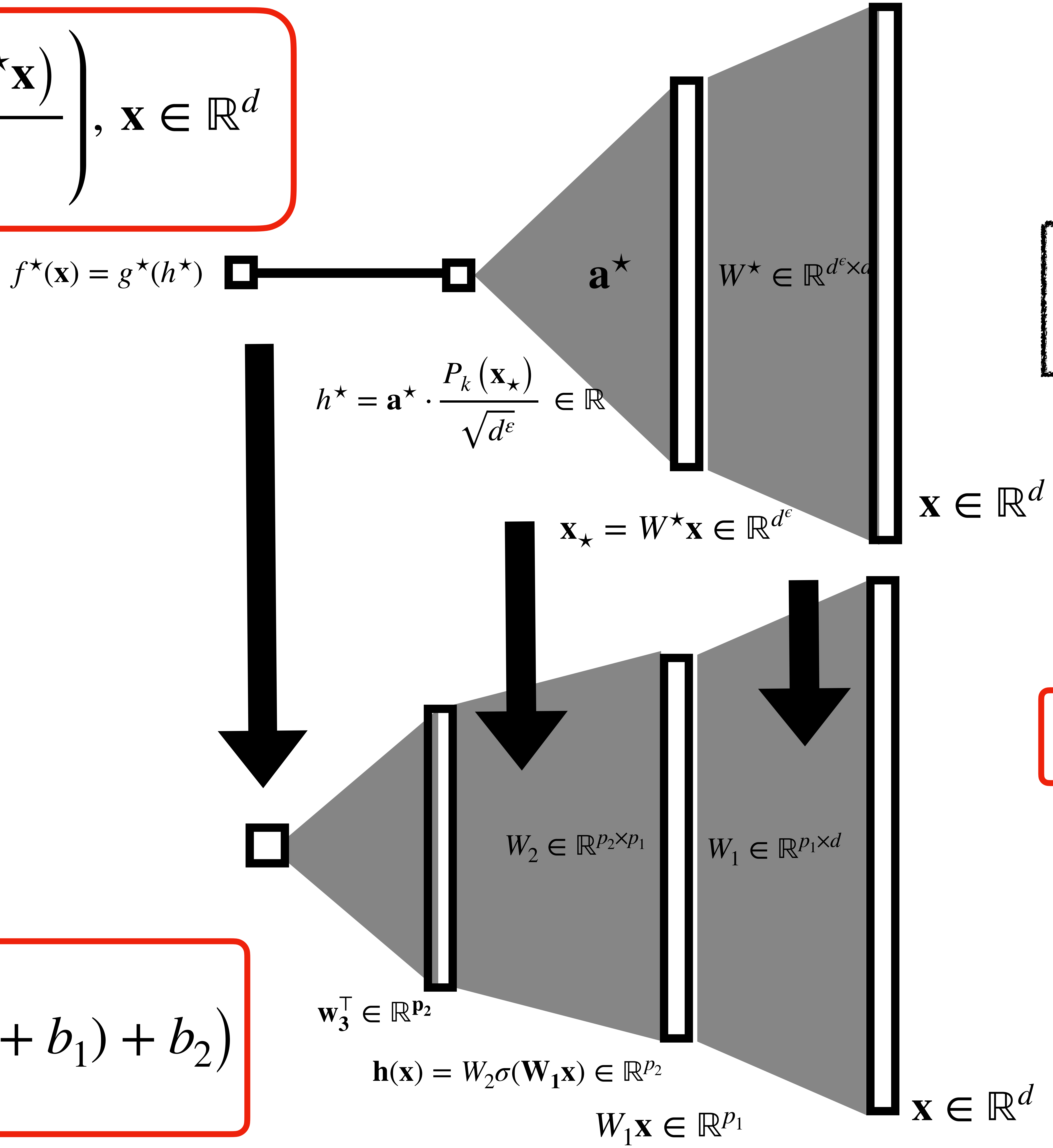

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Expected behavior under idealized scenario

$$f^\star(\mathbf{x}) = g^\star\left(\frac{\mathbf{a}^{\star\top} P_k\left(W^\star \mathbf{x}\right)}{\sqrt{d^{\varepsilon_1}}}\right), \mathbf{x} \in \mathbb{R}^d$$

$$f^\star(\mathbf{x}) = g^\star\left(\frac{\mathbf{a}^{\star\top} P_k\left(x_\star\right)}{\sqrt{d^{\varepsilon}}}\right), \mathbf{x}_\star \in \mathbb{R}^{d^\epsilon}$$

$$\hat{y} = \hat{\mathbf{W}}^3 \sigma(\hat{W}_2 \sigma(\hat{W}_1 \mathbf{x}))$$

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$$\hat{y} = \hat{\mathbf{W}}^3 \sigma(\hat{W}_2 \sigma(\hat{W}_1 \mathbf{x}))$$

With $n = \mathcal{O}(d^{1+\epsilon})$ **data,** $\hat{W}_1$ **can learn the features with GD :** $\hat{W}_1 \approx Z_1 W^\star + Z_2$

$$\hat{y} \approx \hat{\mathbf{w}}_\mathbf{3}{}^\top \sigma(\hat{W}_2 \sigma(Z_1 x_\star + Z_2))$$

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$$\approx \hat{\mathbf{w}}_3{}^\top \sigma(Z_1 h^\star + Z_2)$$

Expected behavior under idealized scenario

$$f^\star(\mathbf{x}) = g^\star\left(\frac{\mathbf{a}^{\star\top} P_k\left(W^\star \mathbf{x}\right)}{\sqrt{d^{\varepsilon_1}}}\right), \ \mathbf{x} \in \mathbb{R}^d$$

$$f^\star(\mathbf{x}) = g^\star\left(h^\star\right), h^\star = \mathbf{a}^\star \cdot \frac{P_k\left(\mathbf{x}_\star\right)}{\sqrt{d^{\varepsilon}}} \in \mathbb{R}$$

$$\hat{y} = \hat{\mathbf{W}}^3 \sigma(\hat{W}_2 \sigma(\hat{W}_1 \mathbf{x}))$$

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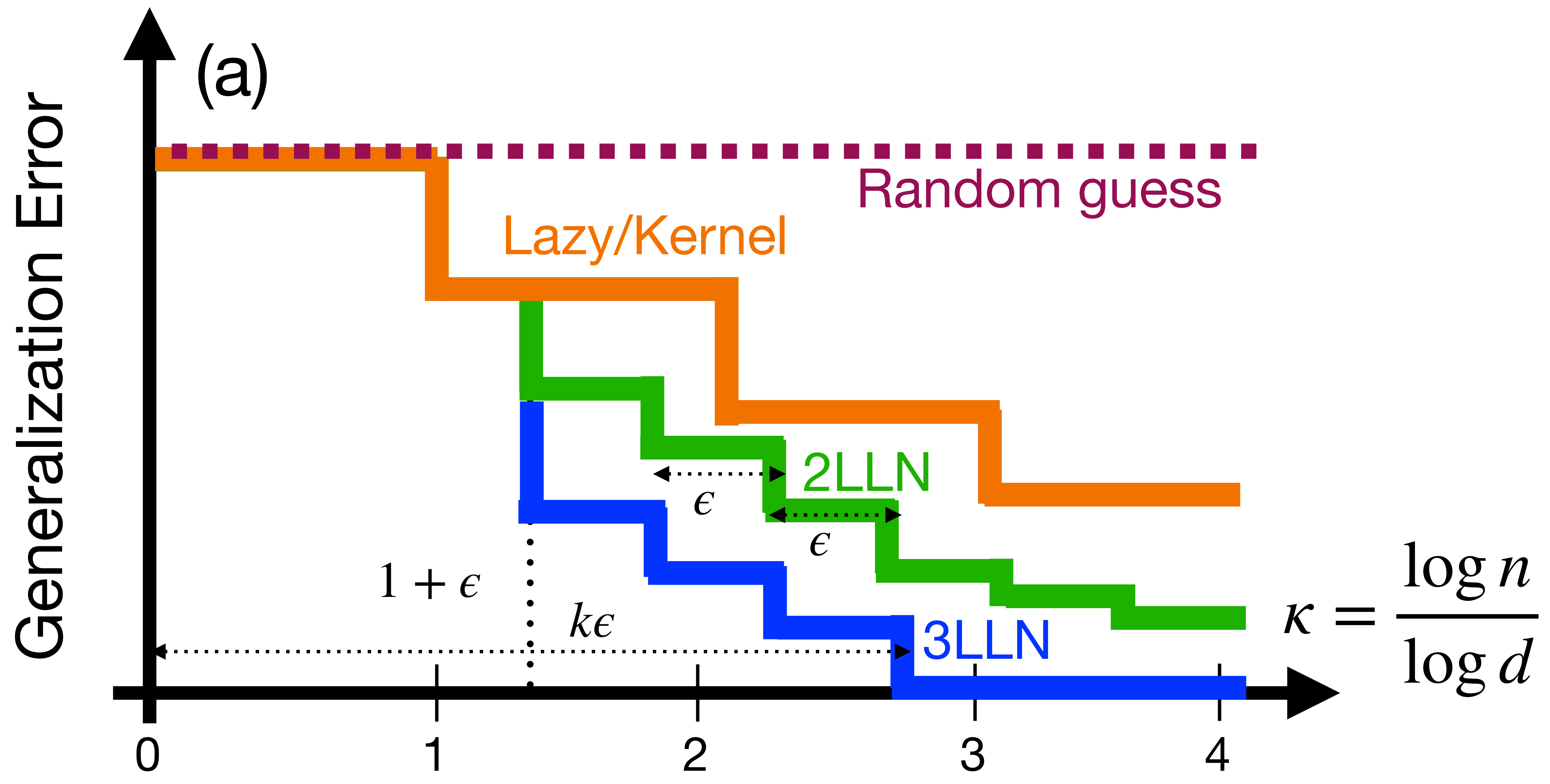
Two levels of dimension reduction



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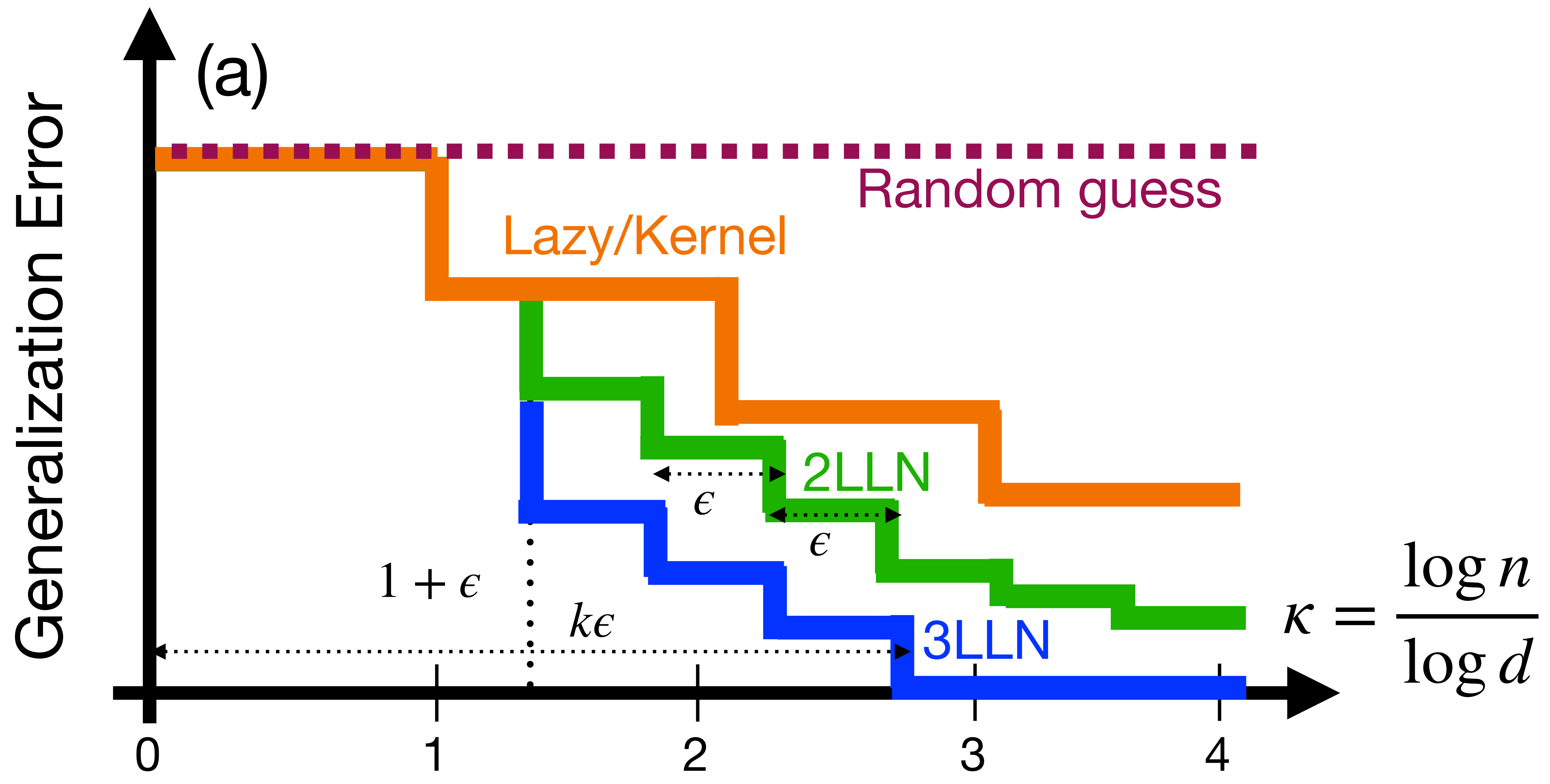
$$f^\star(\mathbf{x}) = \mathbf{g}^\star\left(\frac{\mathbf{a}^{\star\top}\mathbf{P}_\mathbf{k}\left(\mathbf{W}^\star\mathbf{x}\right)}{\sqrt{\mathbf{d}^\varepsilon}}\right), \mathbf{x} \in \mathbb{R}^\mathbf{d}$$



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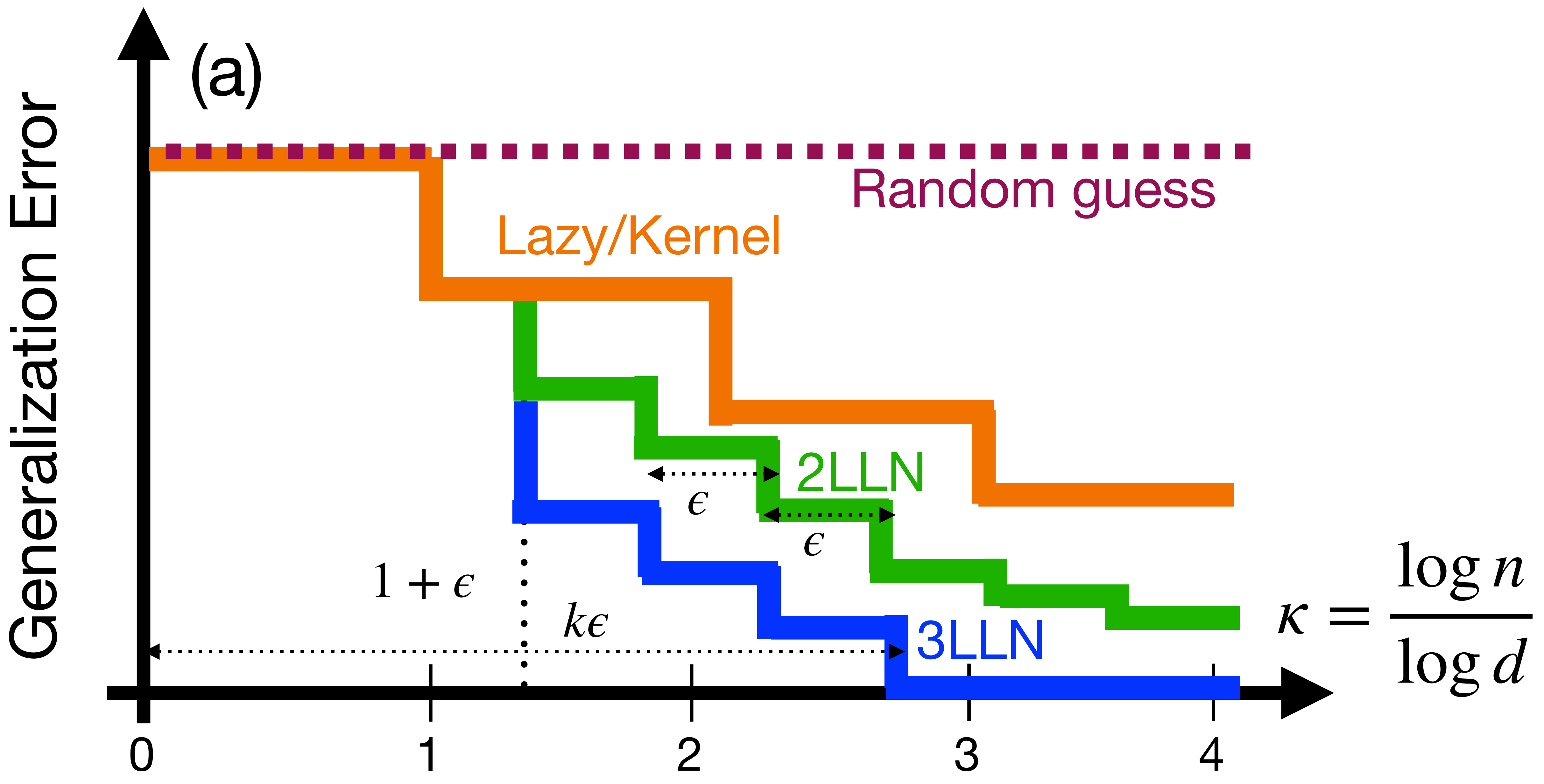

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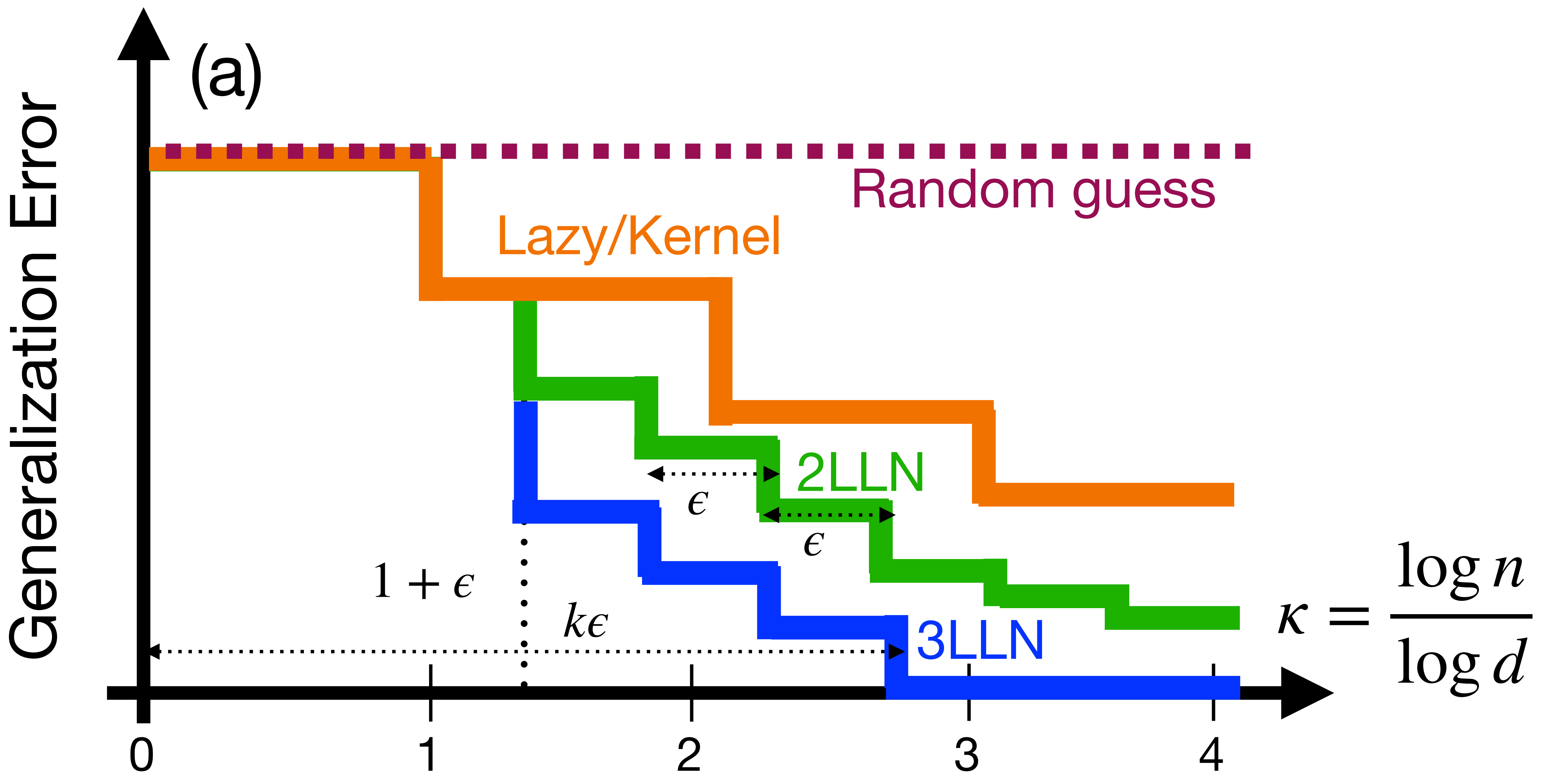
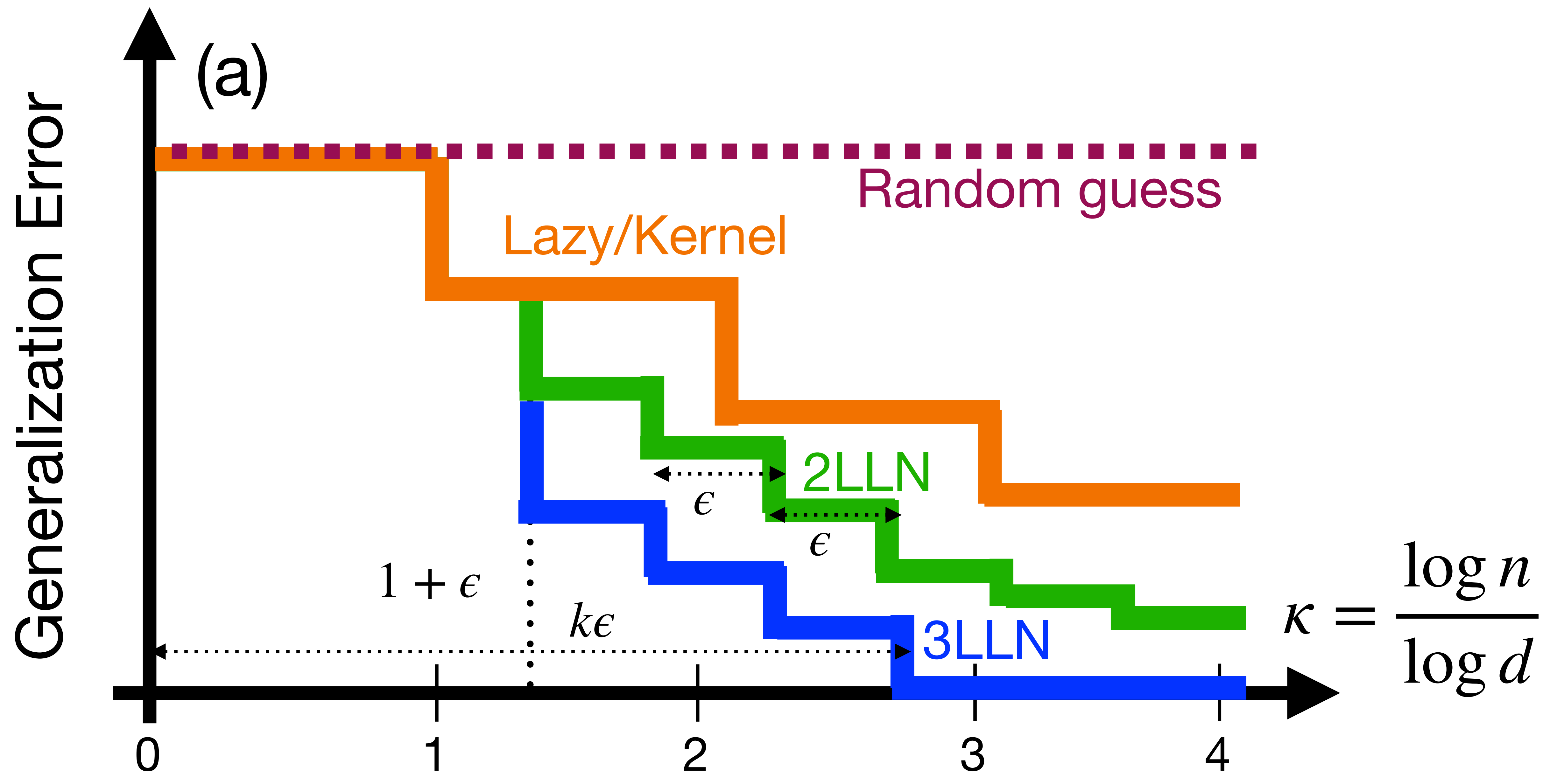


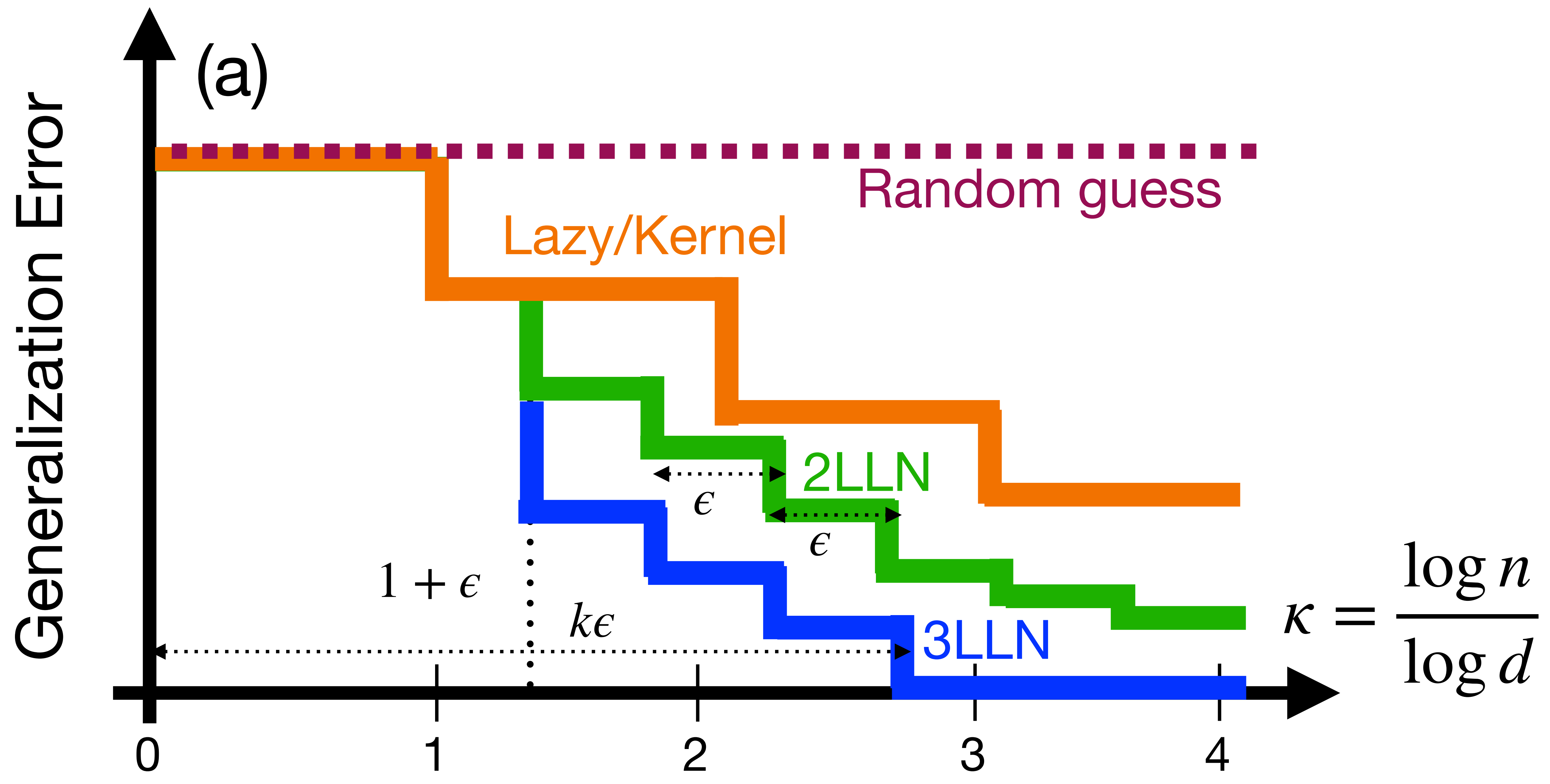
Two levels of dimension reduction

$$f^\star(\mathbf{x}) = \mathbf{g}^\star\left(\frac{\mathbf{a}^{\star\top}\mathbf{P_k}\left(\mathbf{W}^\star\mathbf{x}\right)}{\sqrt{\mathbf{d}^\varepsilon}}\right),\ \mathbf{x} \in \mathbb{R}^{\mathbf{d}} \Rightarrow f^\star(\mathbf{x}) = \mathbf{g}^\star\left(\frac{\mathbf{a}^{\star\top}\mathbf{P_k}\left(\mathbf{x}_\star\right)}{\sqrt{\mathbf{d}^\varepsilon}}\right),\ \mathbf{x}_\star \in \mathbb{R}^{\mathbf{d}_\epsilon} \Rightarrow$$



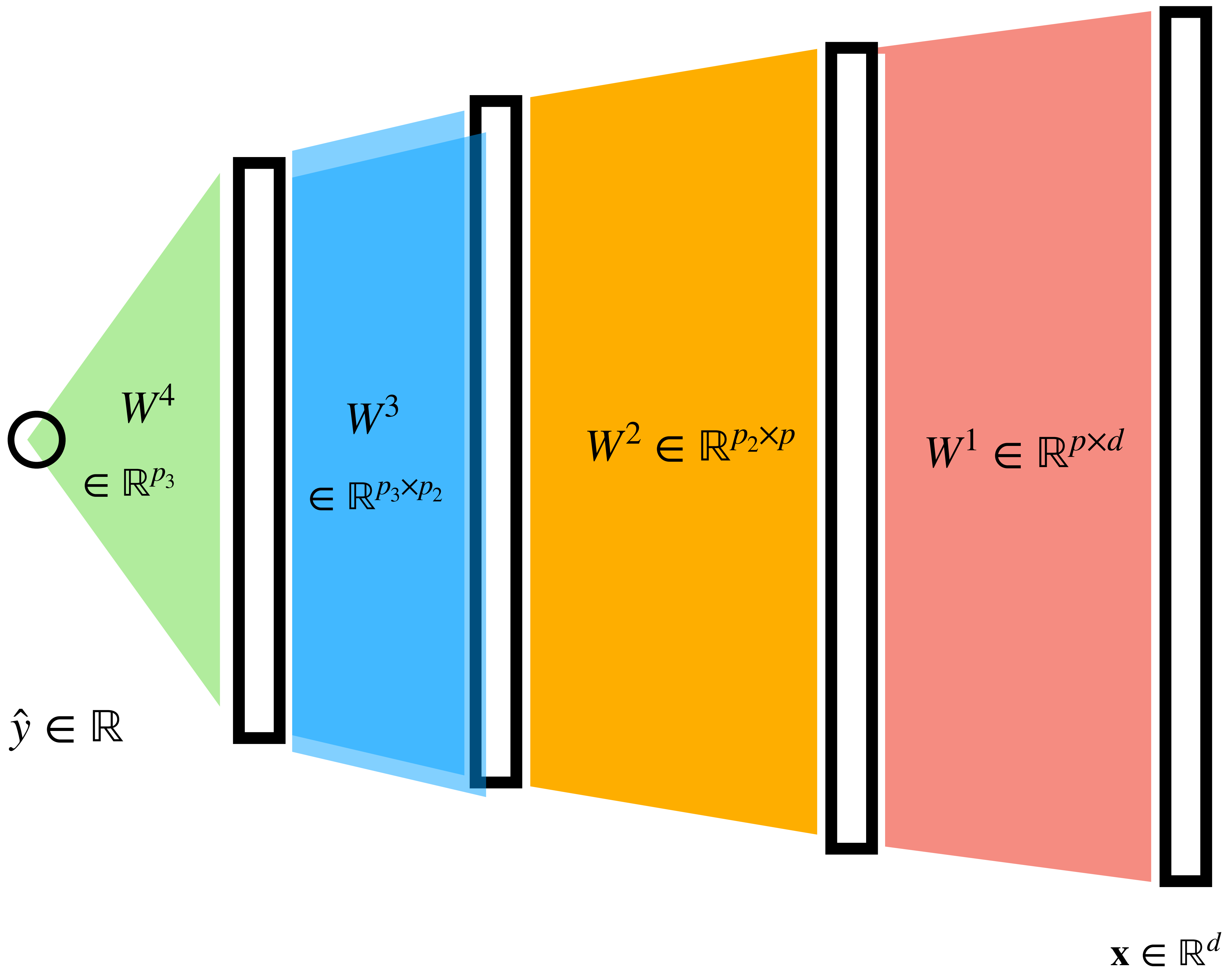
Two levels of dimension reduction

$$f^\star(\mathbf{x}) = \mathbf{g}^\star\left(\frac{\mathbf{a}^{\star\top}\mathbf{P}_\mathbf{k}\left(\mathbf{W}^\star\mathbf{x}\right)}{\sqrt{\mathbf{d}^\varepsilon}}\right), \ \mathbf{x} \in \mathbb{R}^\mathbf{d} \quad \Rightarrow \quad f^\star(\mathbf{x}) = \mathbf{g}^\star\left(\frac{\mathbf{a}^{\star\top}\mathbf{P}_\mathbf{k}\left(\mathbf{x}_\star\right)}{\sqrt{\mathbf{d}^\varepsilon}}\right), \ \mathbf{x}_\star \in \mathbb{R}^{\mathbf{d}_\epsilon} \quad \Rightarrow \quad f^\star(\mathbf{x}) = g^\star\left(h^\star\right)$$

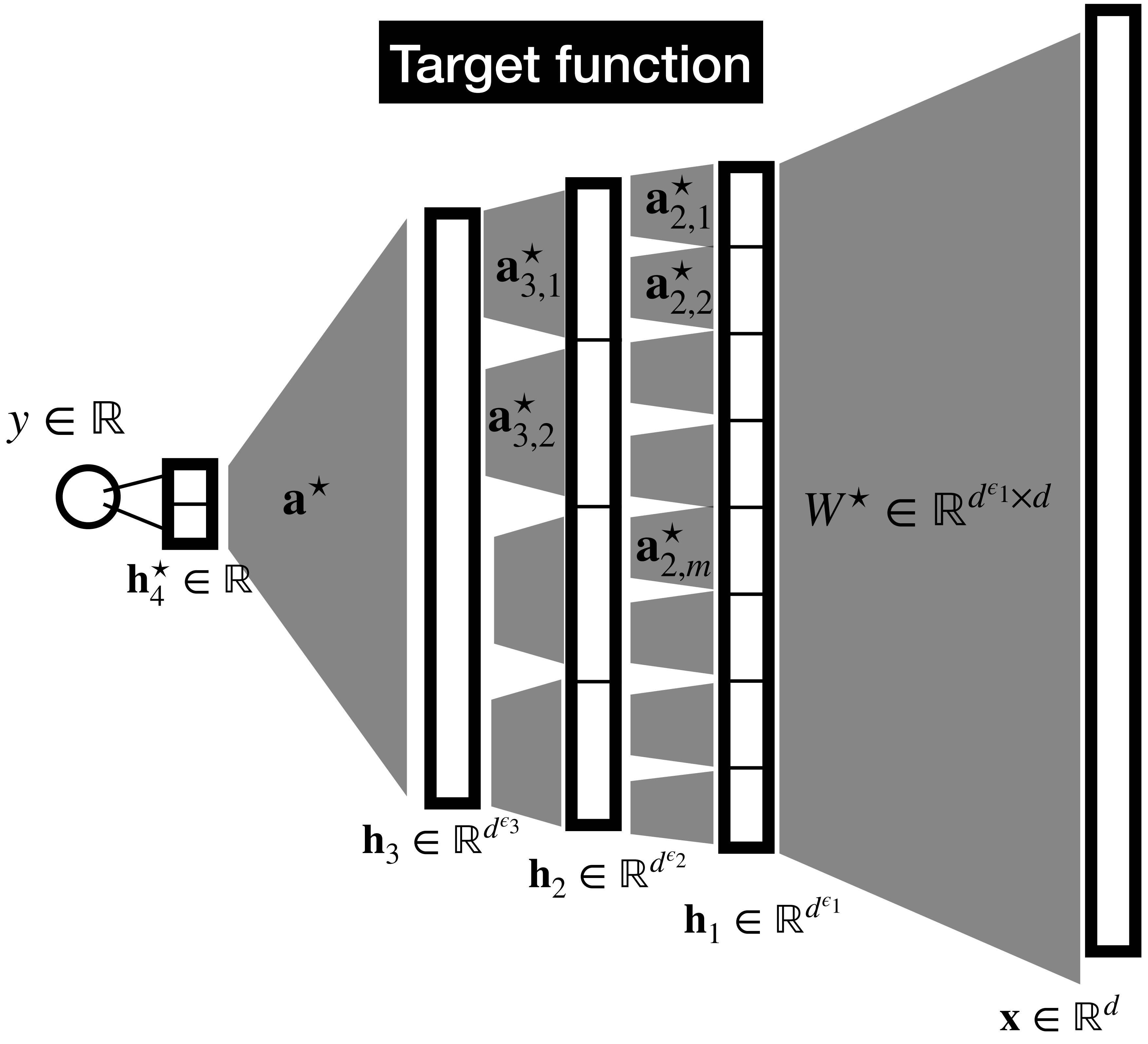


Deep version

Neural network



Target function



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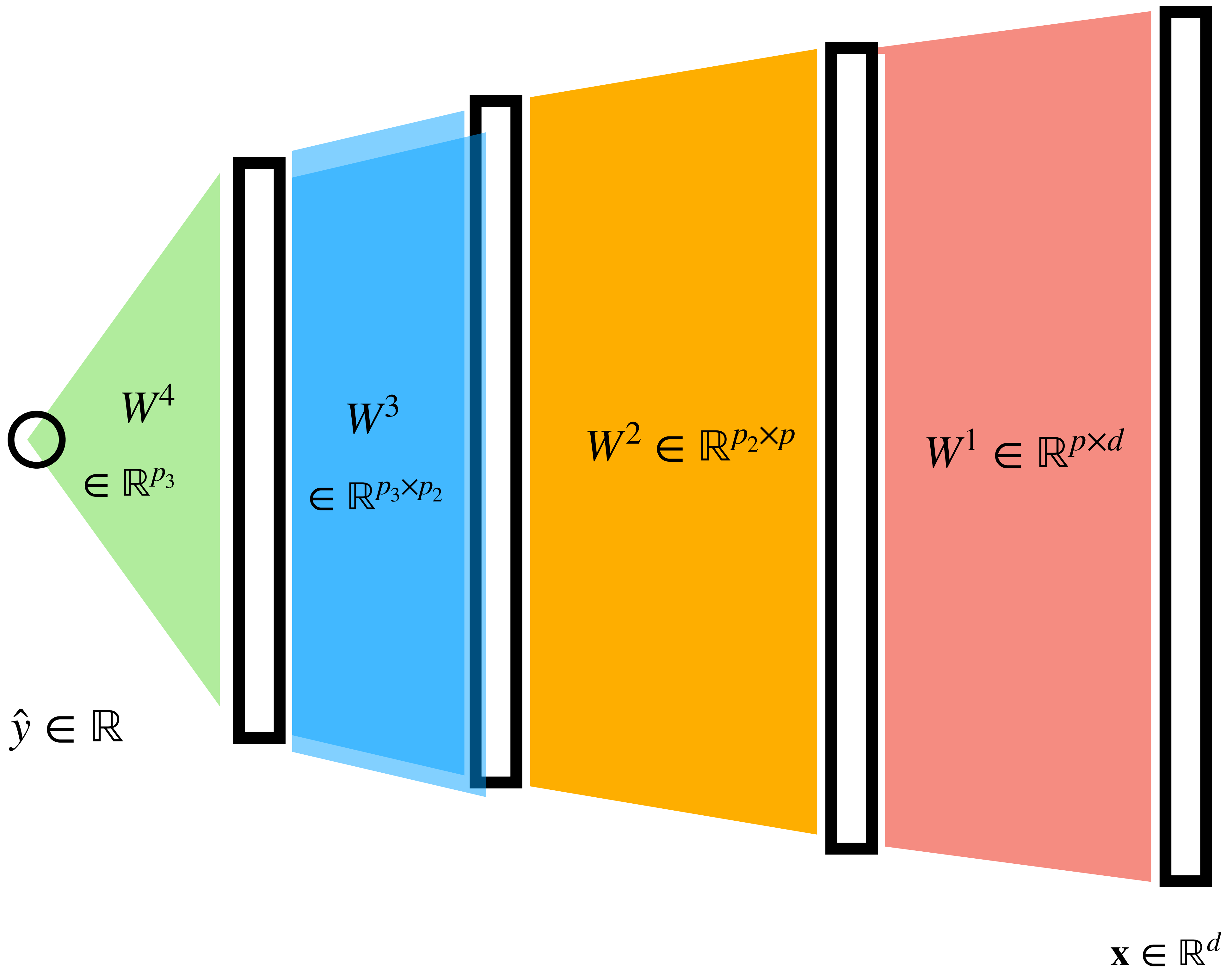

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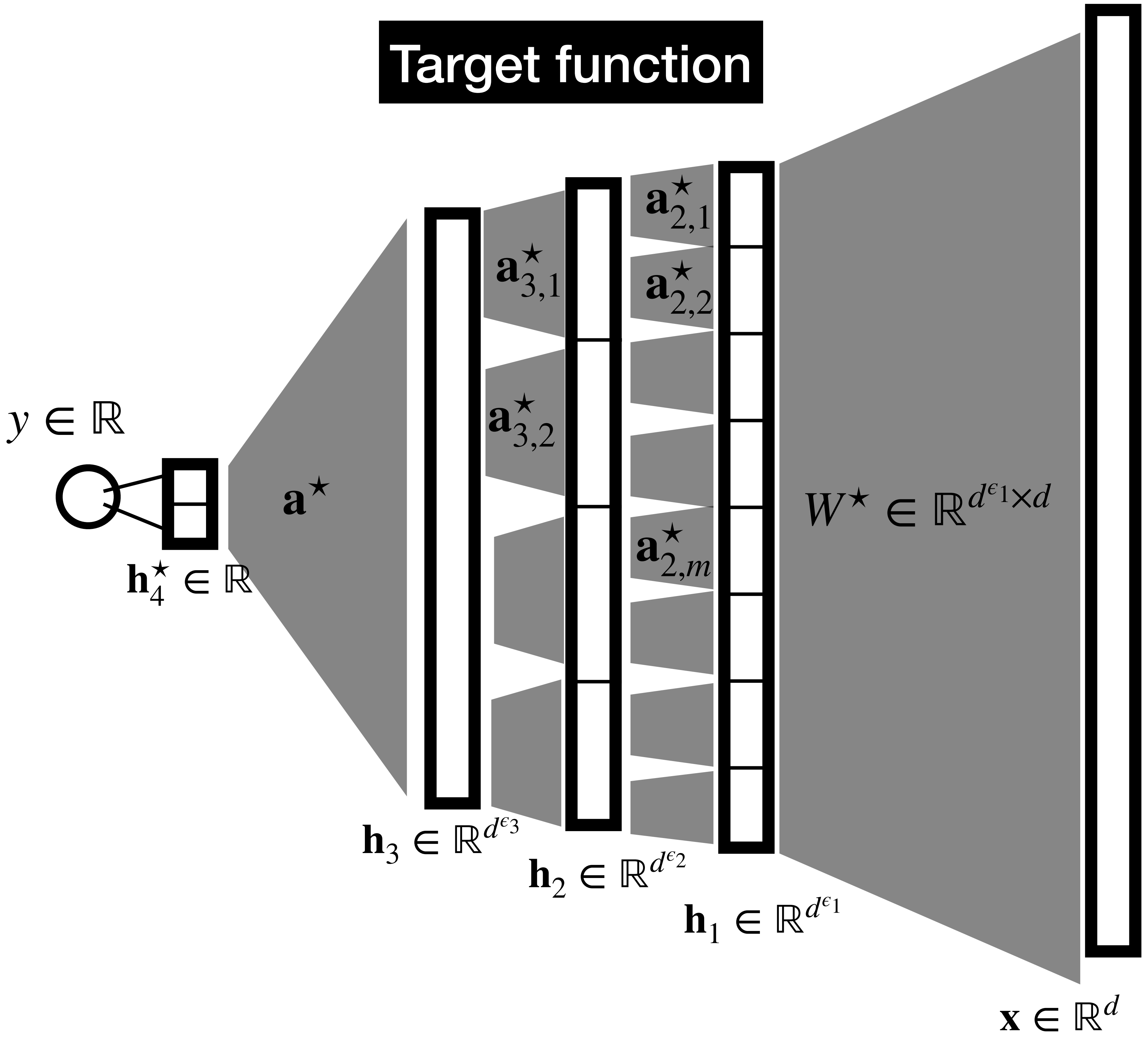
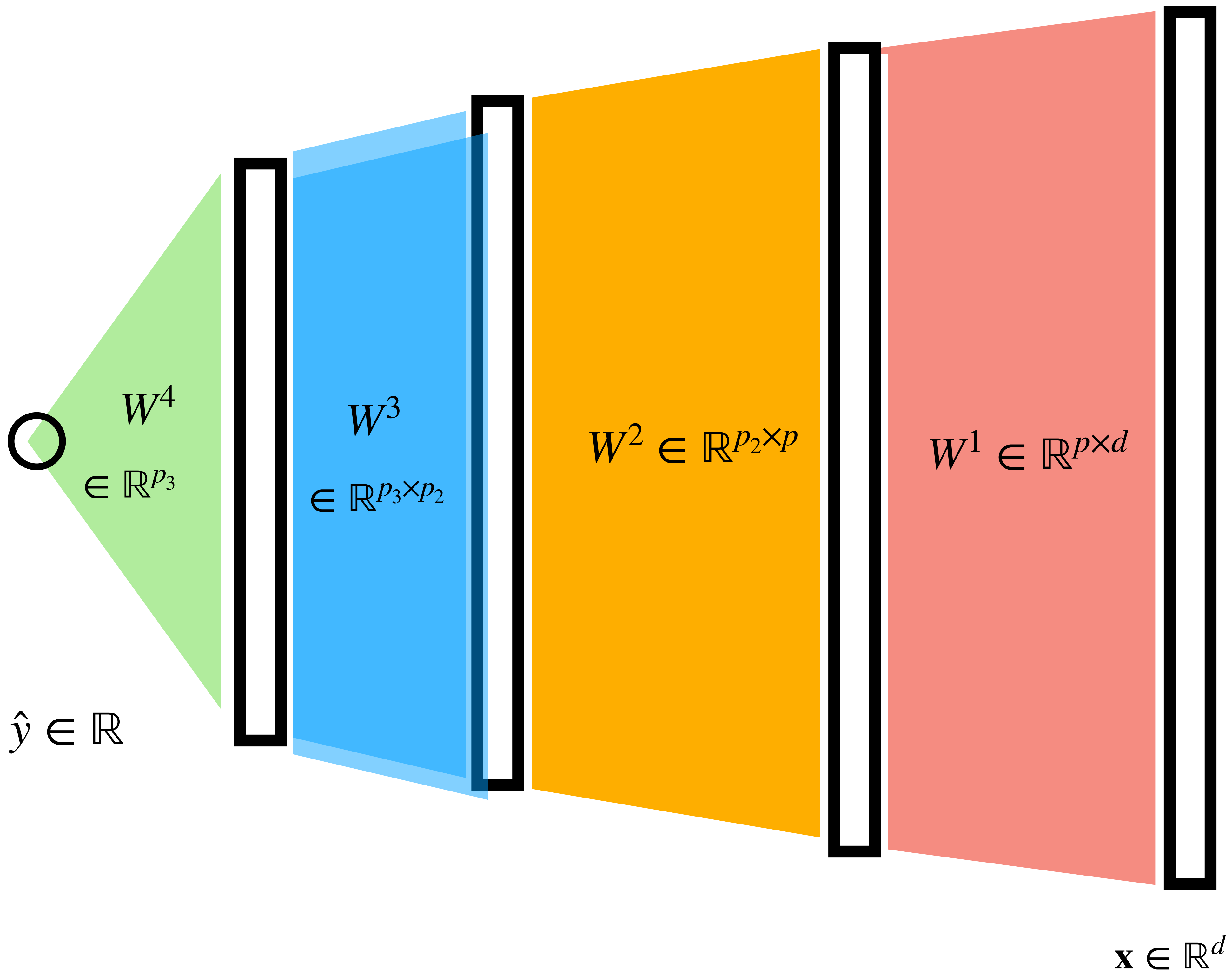


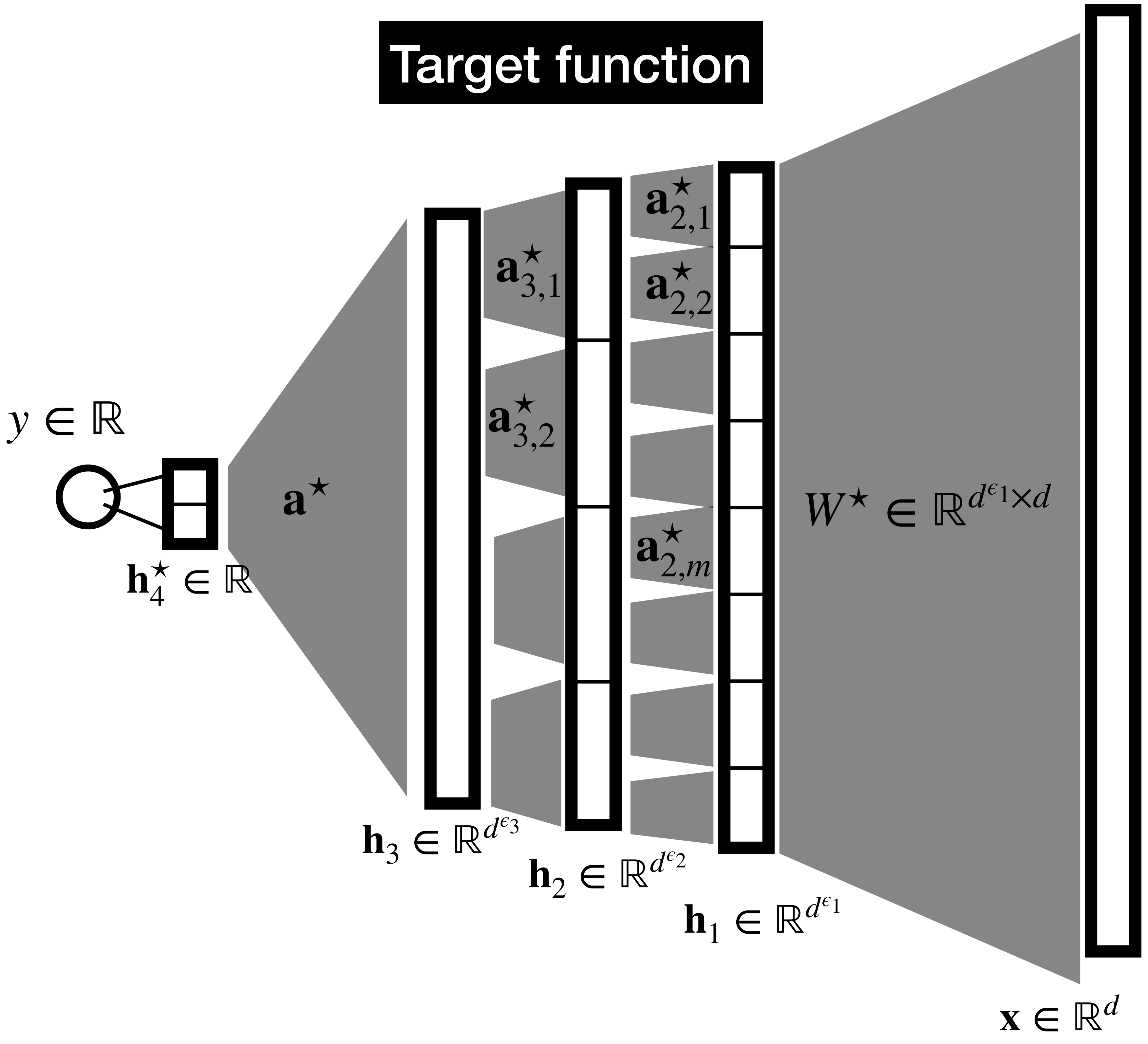
Tree-structure maintains independence of features

Deep version

Neural network



Target function

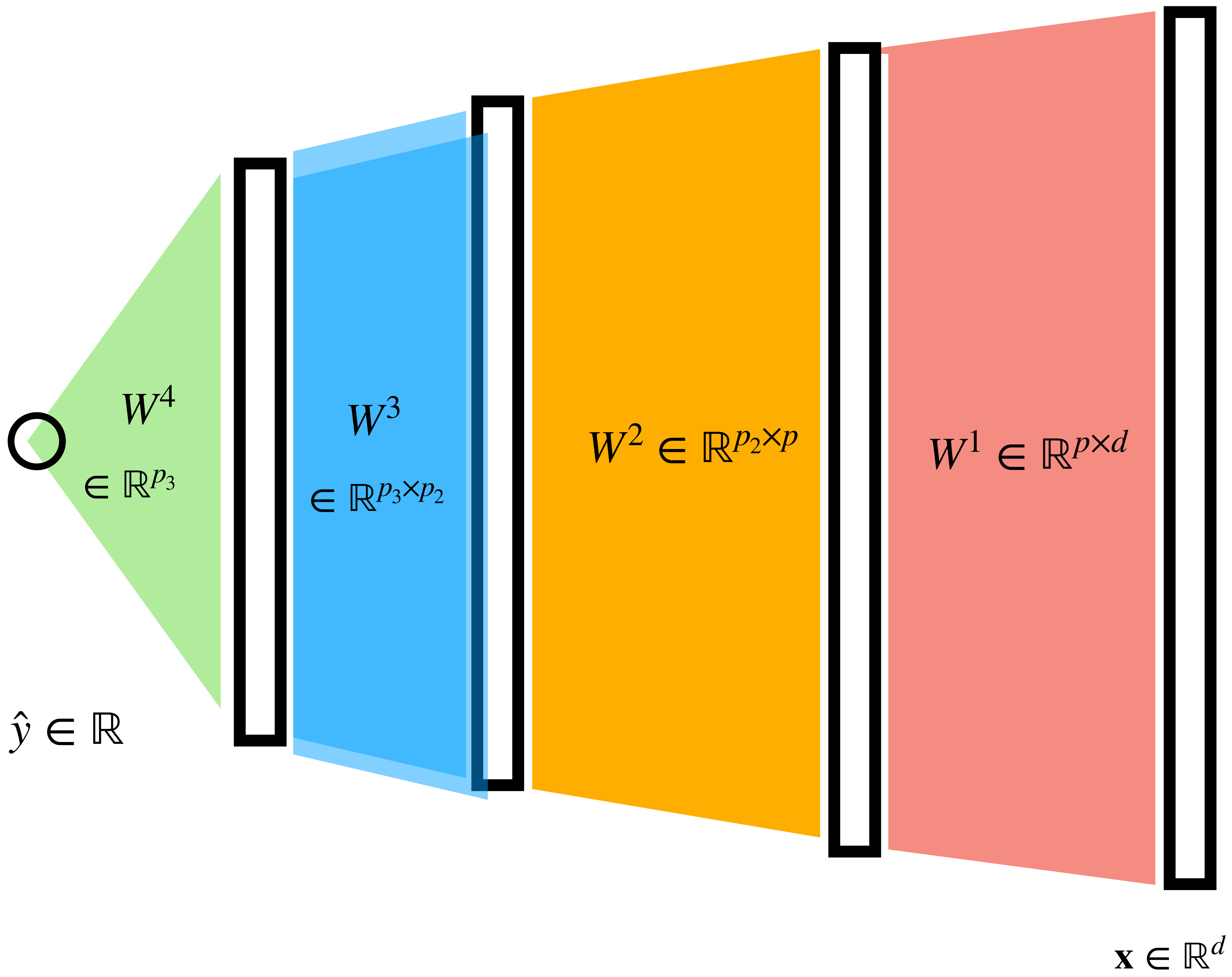


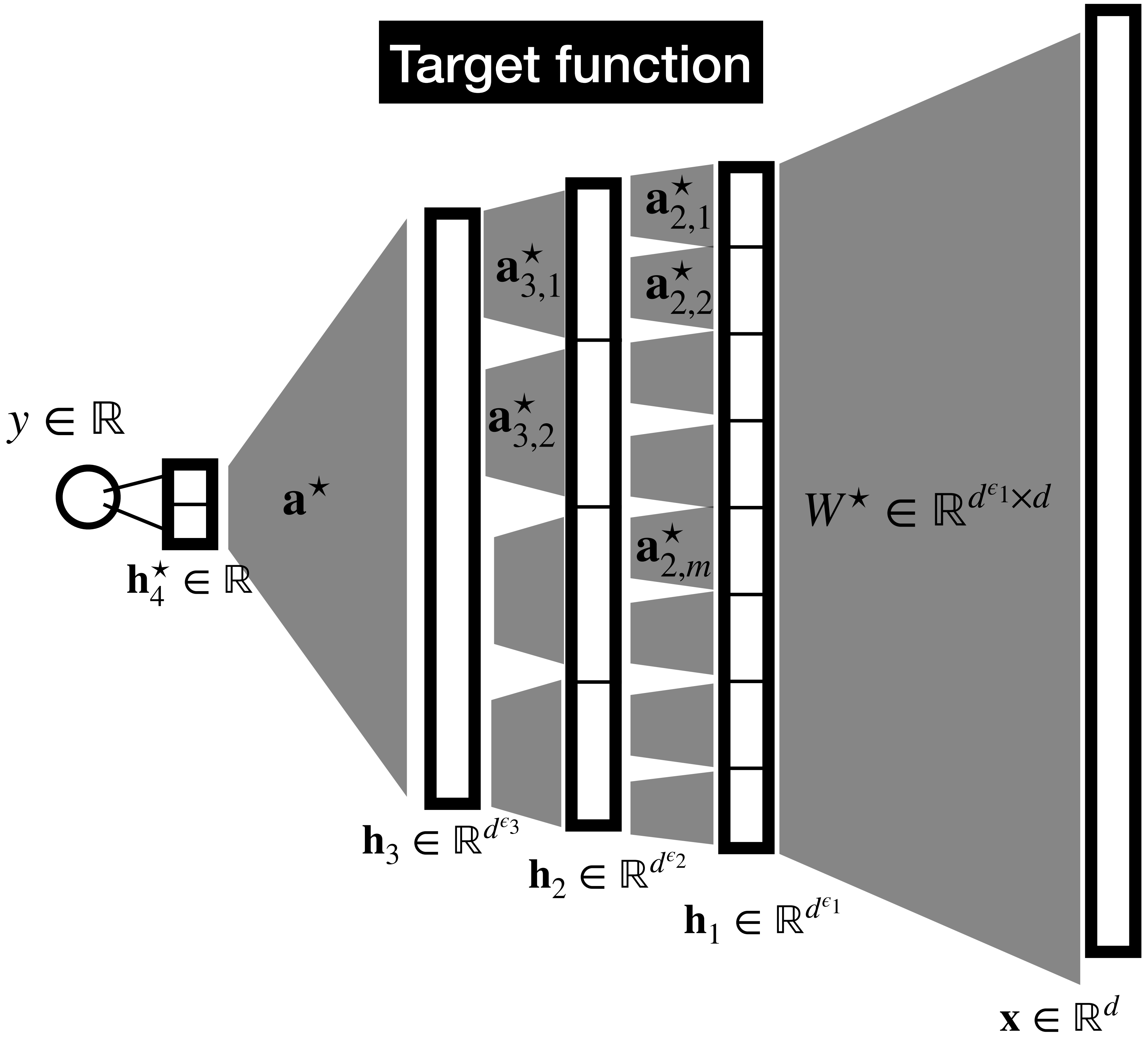
Iterative dimensionality reduction

$$d^{\epsilon_1} \to d^{\epsilon_2} \to d^{\epsilon_3}, \cdots, \to 1$$

Tree-structure maintains independence of features

Deep version





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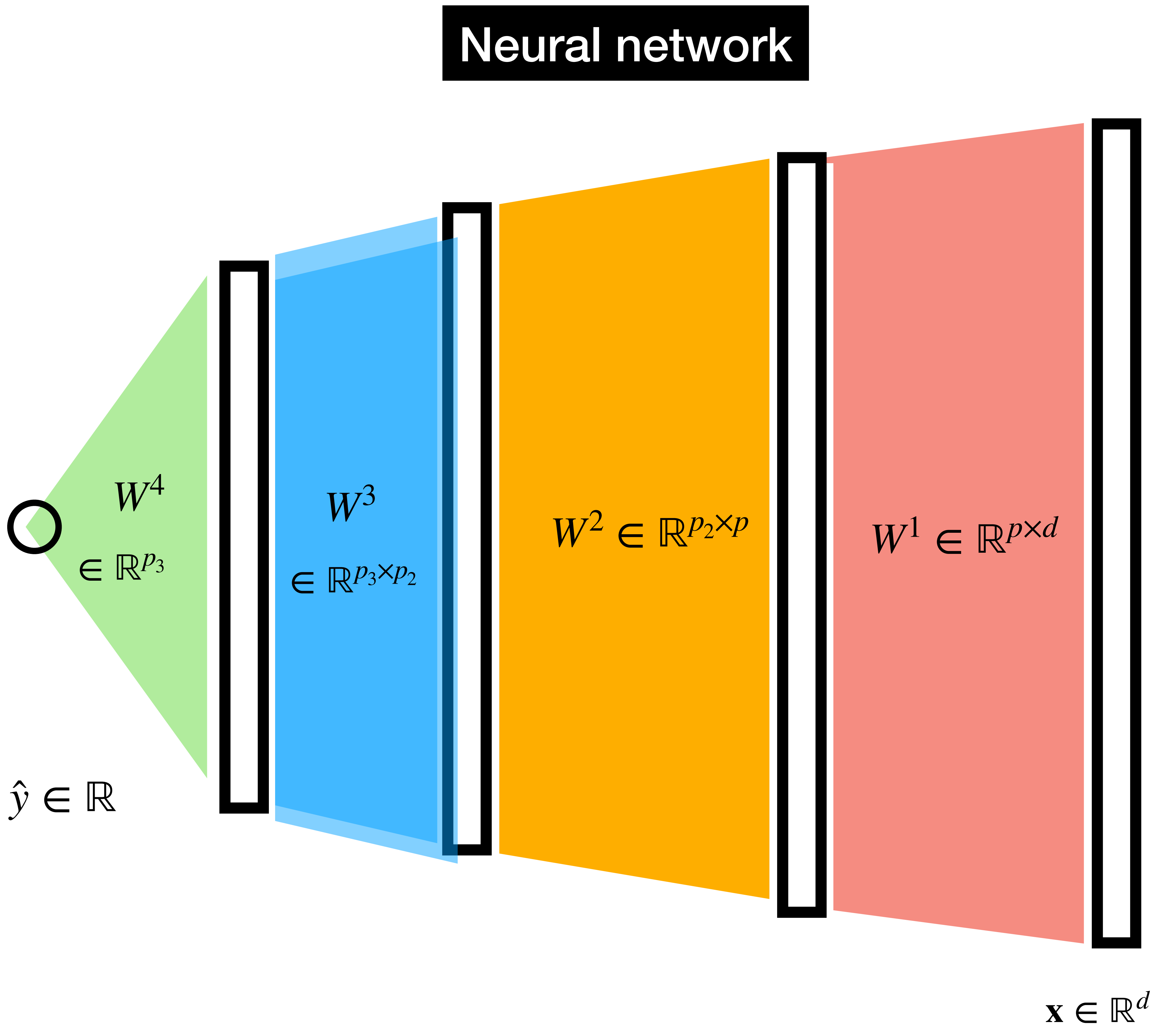
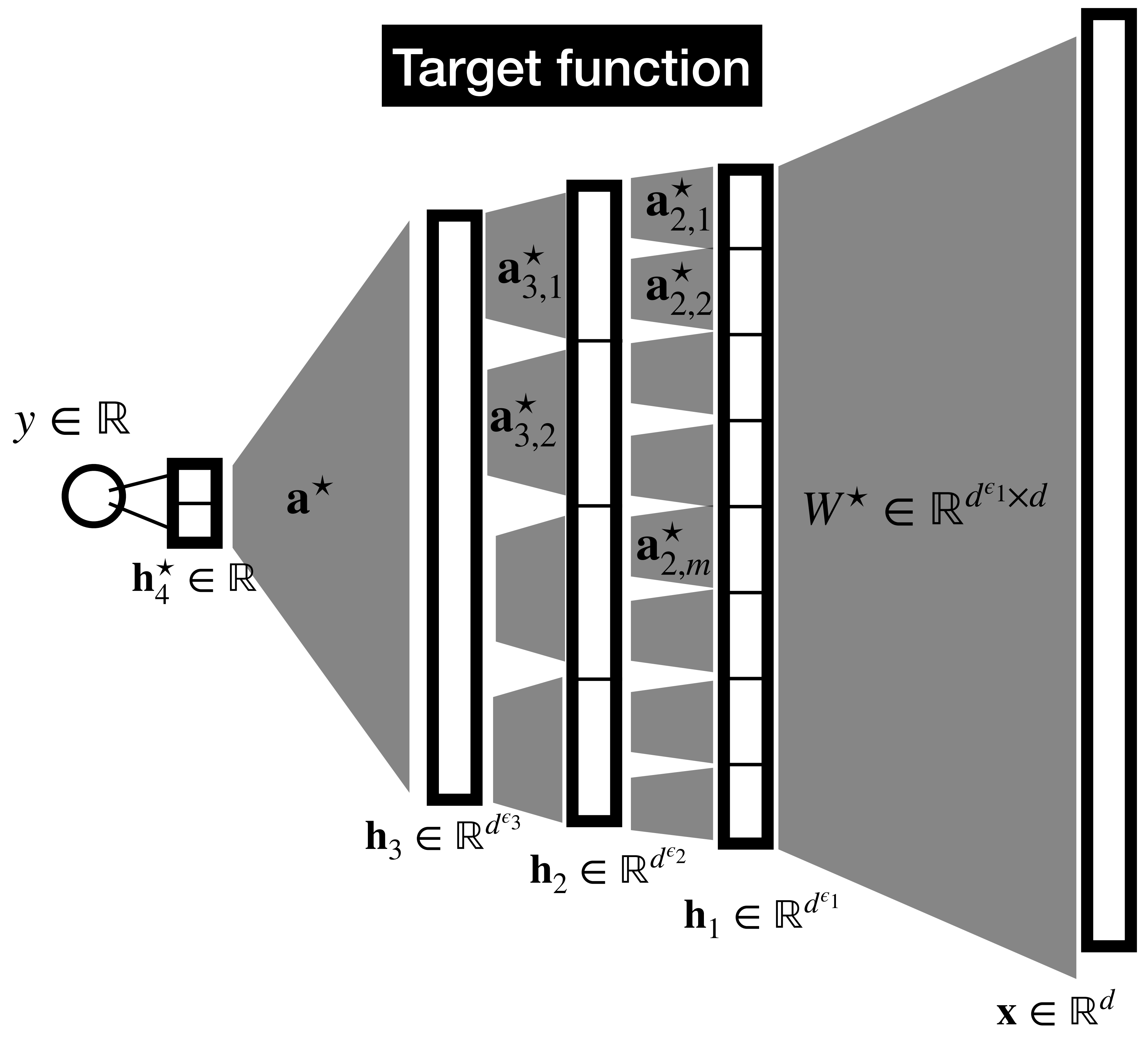
L-layer network

Deep version





Iterative dimensionality reduction

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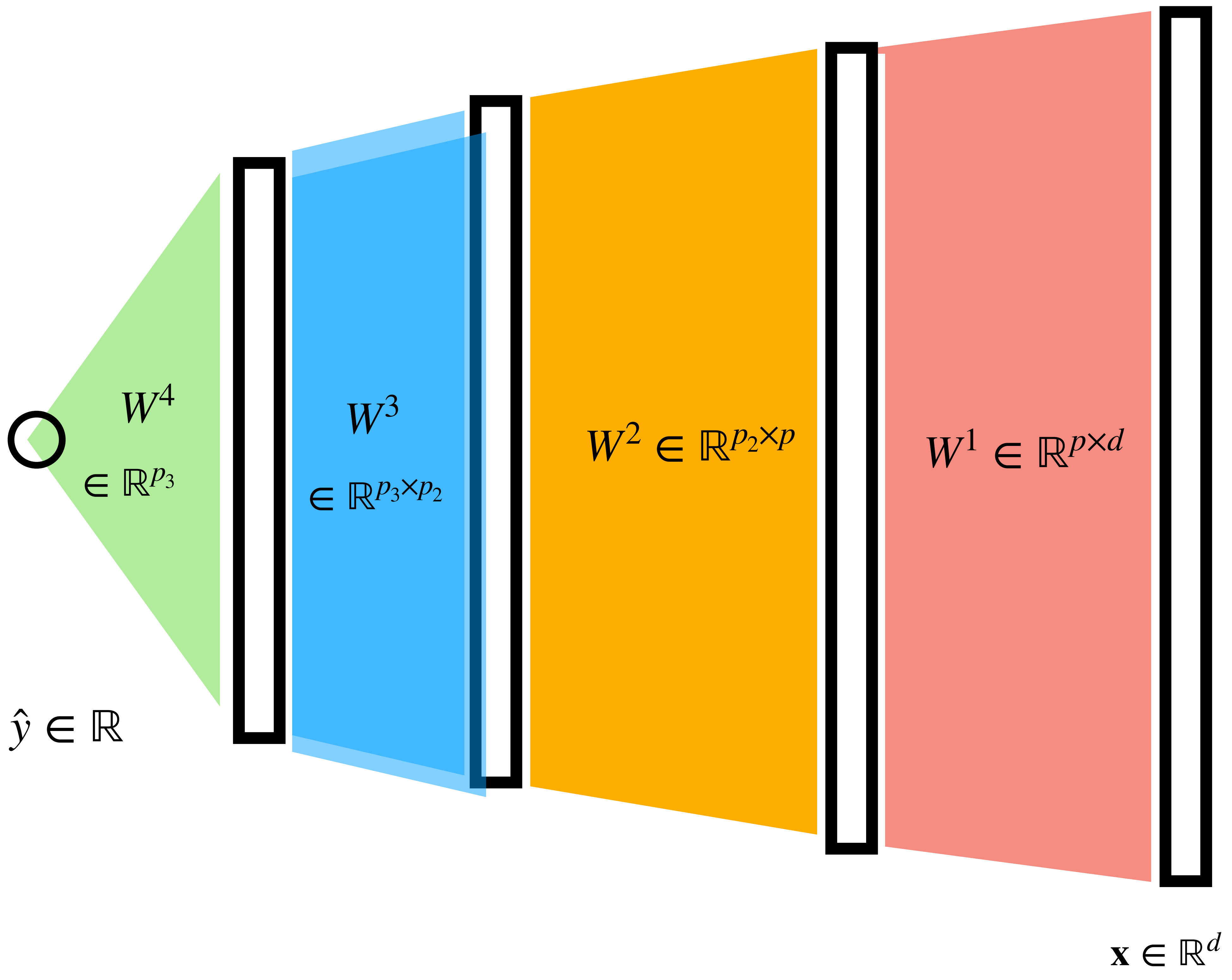
Tree-structure maintains independence of features

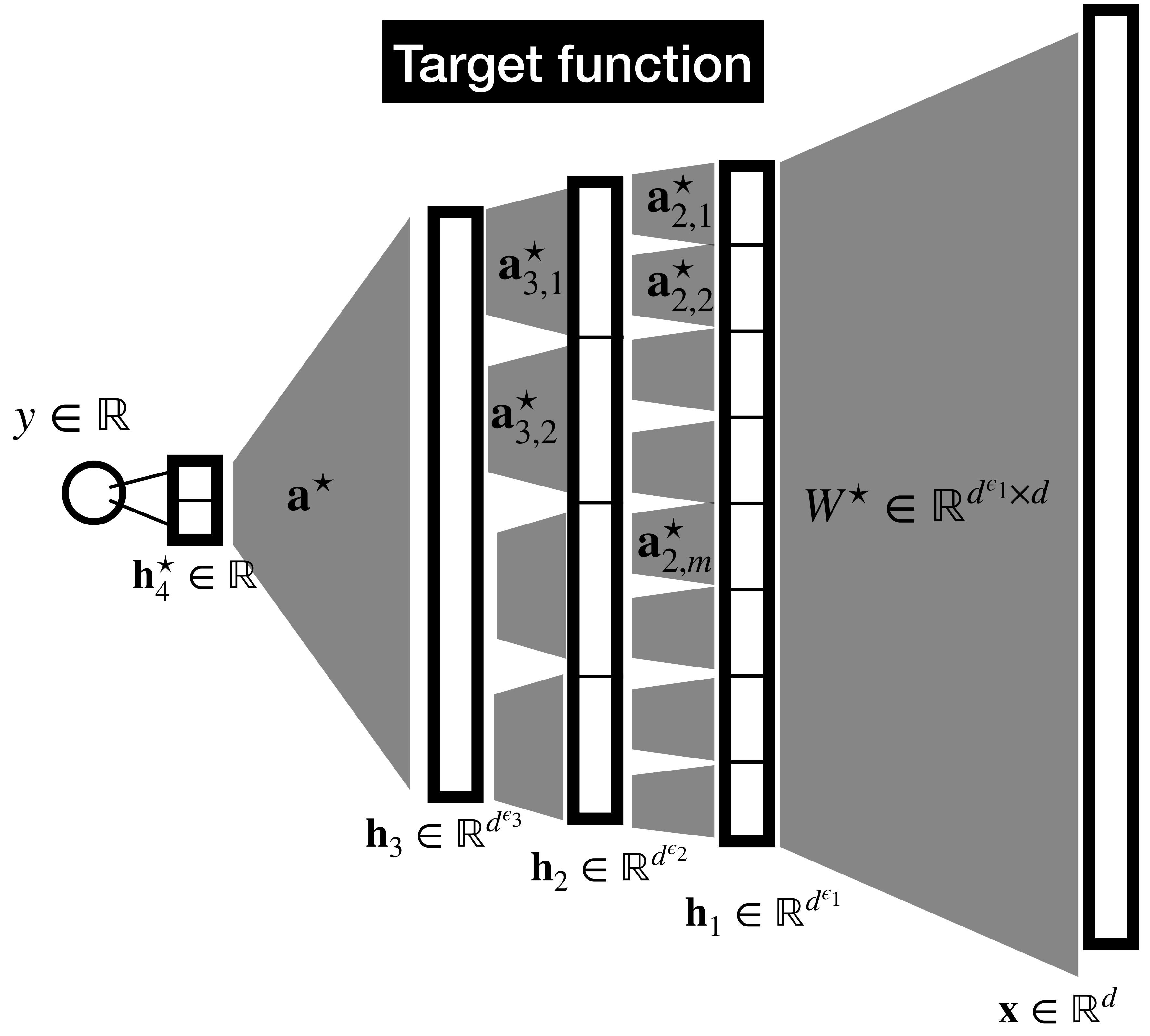
L-layer network →

Deep version

Neural network



Target function



Iterative dimensionality reduction

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Tree-structure maintains independence of features

L-layer network → **L-1 levels of dimension reduction**

Conditions for learnability

Zeiler & Fergus, ***Visualizing and Understanding Convolutional Networks*****, ECCV 2014**

Conditions for learnability

Information exponent

$$\mathrm{IE}(\ell) = \inf\{k : \|\mathbb{E}[(\mathbf{x})^{\otimes k} f^{\star}(\mathbf{x})]\|_F \neq 0\}$$

Zeiler & Fergus, ***Visualizing and Understanding Convolutional Networks*****, ECCV 2014**

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Conditions for learnability

Information exponent → **Compositional Information exponent**

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$$\mathrm{CIE}(\ell) = \inf\{k : \|\mathbb{E}[(h_\ell(\mathbf{x}))^{\otimes k} f^\star(\mathbf{x})]\|_F = \Theta(1)\}$$

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Requires non-trivial low-degree correlations between intermediate features and labels

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Matches behavior of real data

	Cal-101 (30/class)	Cal-256 (60/class)
SVM (1)	44.8 ± 0.7	24.6 ± 0.4
SVM (2)	66.2 ± 0.5	39.6 ± 0.3
SVM (3)	72.3 ± 0.4	46.0 ± 0.3
SVM (4)	76.6 ± 0.4	51.3 ± 0.1
SVM (5)	$\mathbf{86.2 \pm 0.8}$	65.6 ± 0.3
SVM (7)	$\mathbf{85.5 \pm 0.4}$	$\mathbf{71.7 \pm 0.2}$
Softmax (5)	82.9 ± 0.4	65.7 ± 0.5
Softmax (7)	$\mathbf{85.4 \pm 0.4}$	$\mathbf{72.6 \pm 0.1}$

Table 7. Analysis of the discriminative information contained in each layer of feature maps within our ImageNet-pretrained convnet. We train either a linear SVM or softmax on features from different layers (as indicated in brackets) from the convnet. Higher layers generally produce more discriminative features.

Zeiler & Fergus, *Visualizing and Understanding Convolutional Networks*, ECCV 2014

Full set of conditions

Essential assumptions	Technical assumptions

Full set of conditions

Essential assumptions

- $g^\star$: information exponent 1.
- P_k: information exponent/leap ≤ 2

$$\mathrm{CIE}(1) = IE(g^\star) \times IE(P_k)$$

- expressive σ (non-zero Hermites) and regular.

Technical assumptions

Full set of conditions

Essential assumptions

- $g^\star$: information exponent 1.
- P_k: information exponent/leap ≤ 2

$$\mathrm{CIE}(1) = IE(g^\star) \times IE(P_k)$$

- expressive σ (non-zero Hermites) and regular.

Technical assumptions

- $a_i^\star = 1 \ \forall i$ (symmetric targets)
- Correlation loss
- Re-initialization of layers
- expressive (non-zero Hermites) and regular.
- P_k: information exponent $\neq 1$(to avoid spikes)
- $\mathbb{E}[\sigma(\sigma(z))\mathrm{He}_2(z)]\mathbb{E}[P_k(z)\mathrm{He}_2(z)] > 0,$ $\mathbb{E}[\sigma(\sigma(z))z] = 0,$
- $\mathbb{E}_{z\sim\mathcal{N}(0,1)}[g^*(z)\mathrm{He}_j(z) = 0, \ 1 < j \leq k]$

Main theorem

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$$\hat{f}(\mathbf{x}) = \mathbf{w_3}^\top \sigma\left(W_2\sigma(W_1\mathbf{x} + b_1) + b_2\right)$$

$$W_1 \in \mathbb{R}^{p_1 \times d}, W_2 \in \mathbb{R}^{p_2 \times p_1}, \mathbf{w_3} \in \mathbb{R}^{\mathbf{p_3}}$$

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Let $\delta > 0$ be arbitrarily small

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- First layer: $\mathcal{O}(\log d)$ steps of spherical SGD on W_1 with correlation loss, batch-size $\mathcal{O}(d^{1+\epsilon+\delta})$, vanishing step-size:

Main theorem

$$\hat{f}(\mathbf{x}) = \mathbf{w_3}^\top \sigma\left(W_2\sigma(W_1\mathbf{x} + b_1) + b_2\right)$$

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$$W_1 = Z(W^\star) + o_d(1), z_i \sim U(S_1)$$

Main theorem

$$\hat{f}(\mathbf{x}) = \mathbf{w_3}^\top \sigma\left(W_2 \sigma(W_1 \mathbf{x} + b_1) + b_2\right)$$

$$W_1 \in \mathbb{R}^{p_1 \times d}, W_2 \in \mathbb{R}^{p_2 \times p_1}, \mathbf{w_3} \in \mathbb{R}^{\mathbf{p_3}}$$

Let $\delta > 0$ be arbitrarily small

- First layer: $\mathcal{O}(\log d)$ steps of spherical SGD on W_1 with correlation loss, batch-size $\mathcal{O}(d^{1+\epsilon+\delta})$, vanishing step-size:

$$W_1 = Z(W^\star) + o_d(1), z_i \sim U(S_1)$$

- Second layer: upon reinitializing $W_2 = \mathbf{0}_{d \times d}$, one step of pre-conditioned step with batch-size $\mathcal{O}(d^{k\epsilon+\delta}),\ p_1 = \mathcal{O}(d^{k\epsilon+\delta})$:

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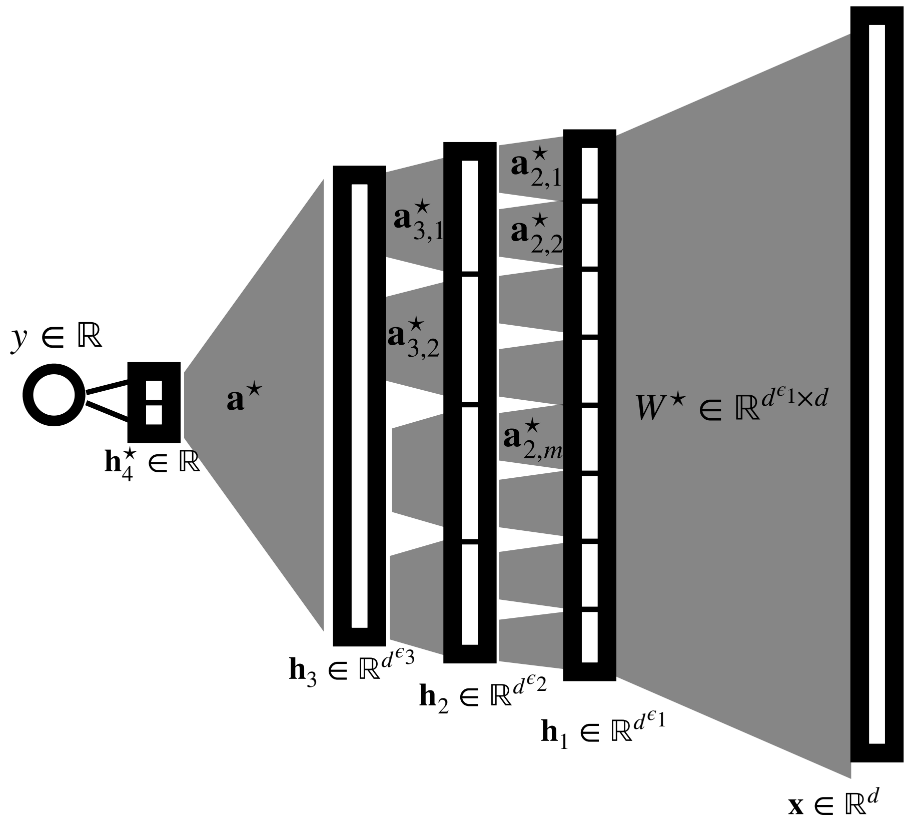

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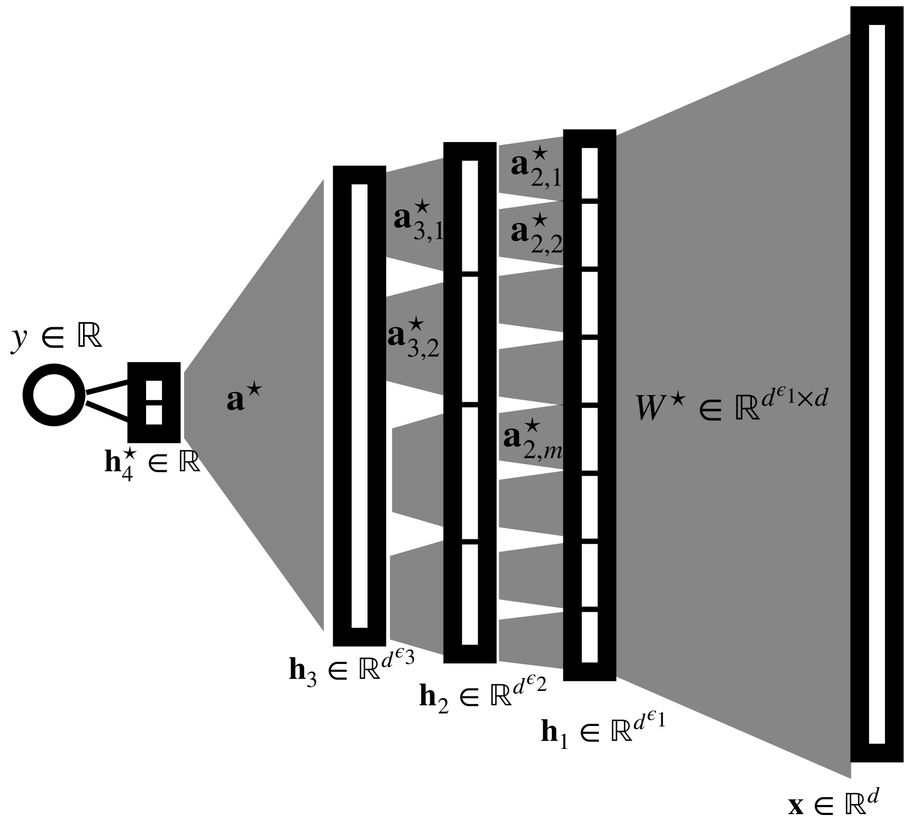

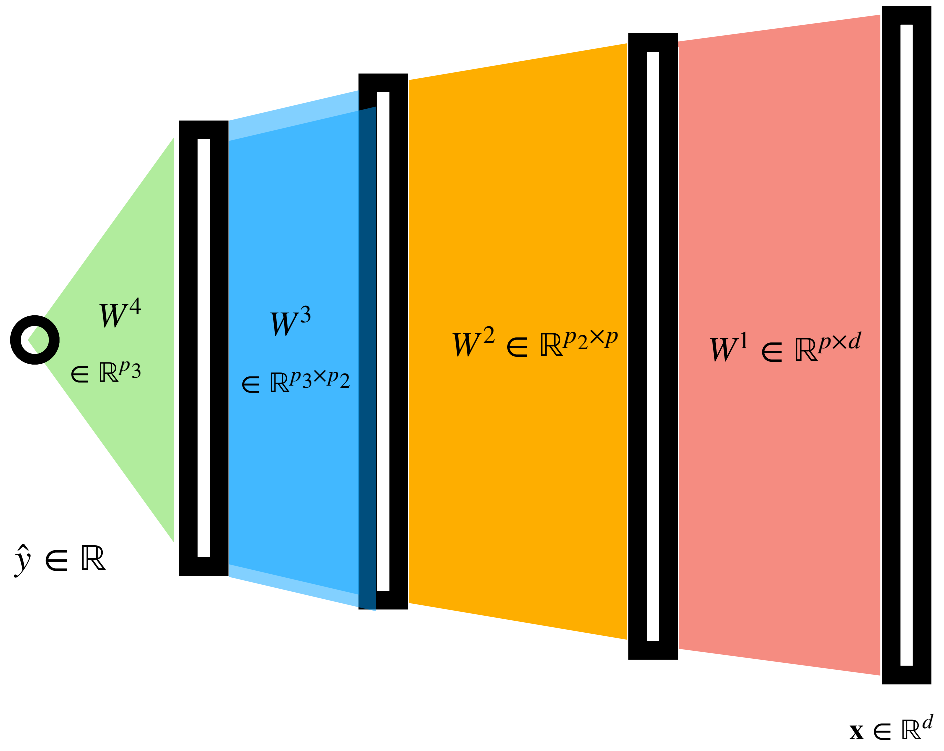

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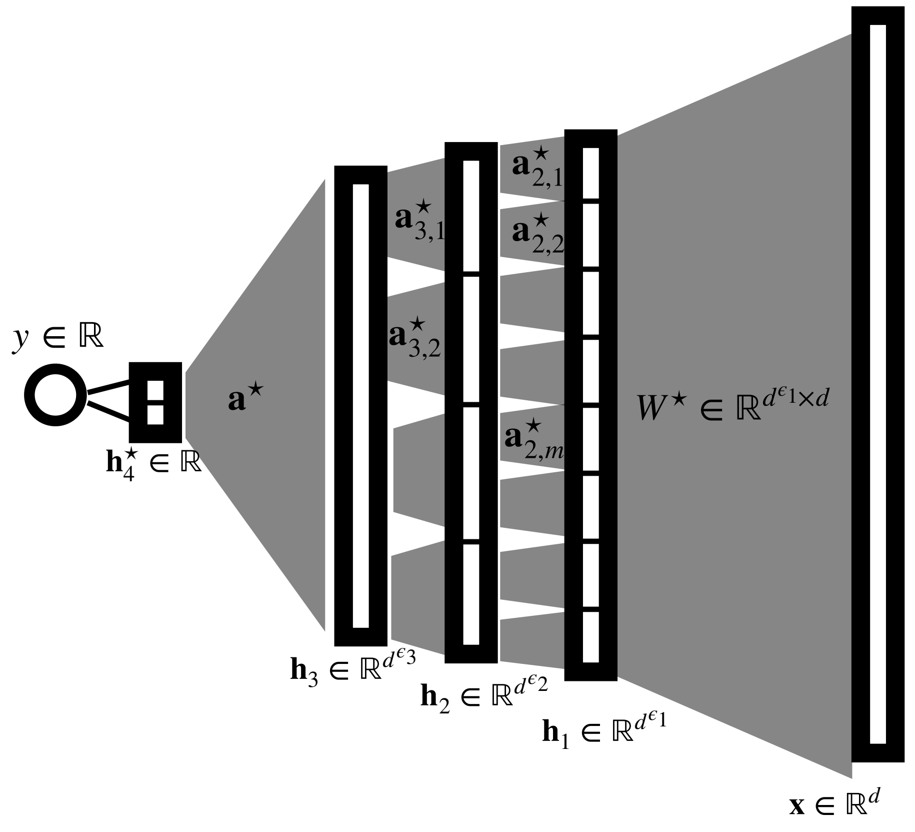

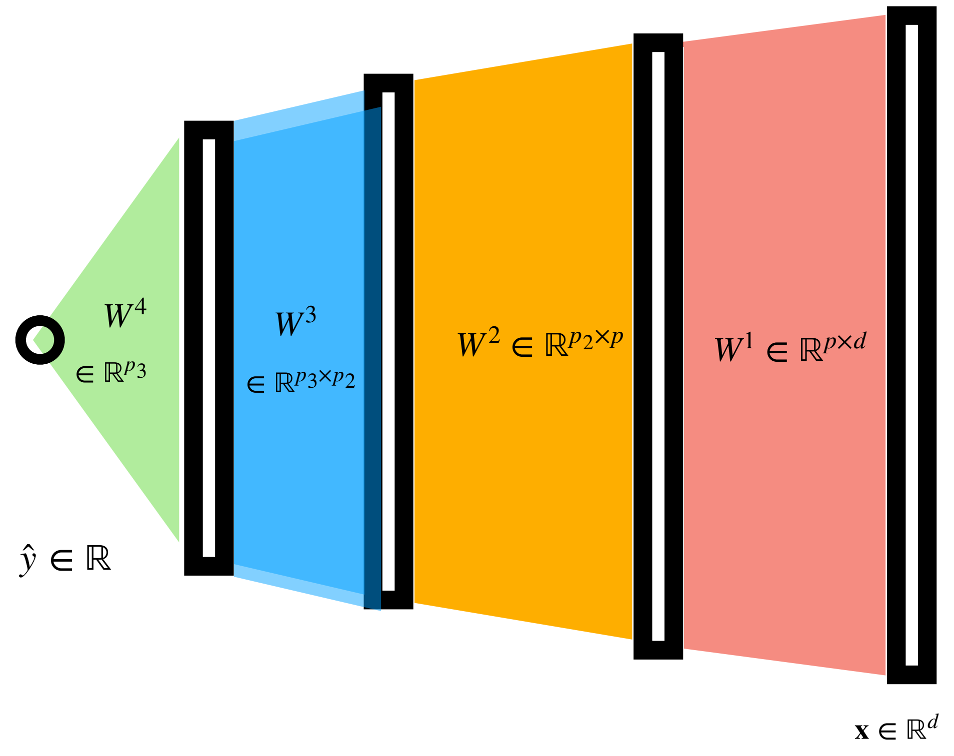

- Conditional on perfect spherical recovery for W_{L-2}, the same picture holds for the last two layers.

- Key idea: Features are independent by tree-structure, asymptotically Gaussian, and maintain nice tails (hypercontractivity).

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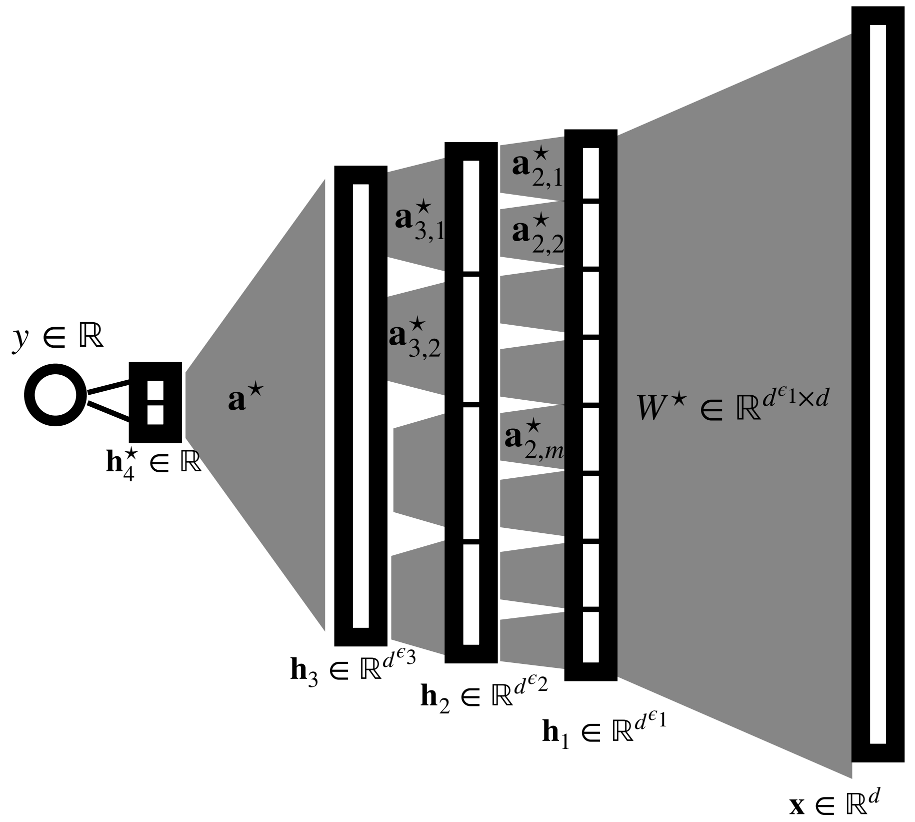

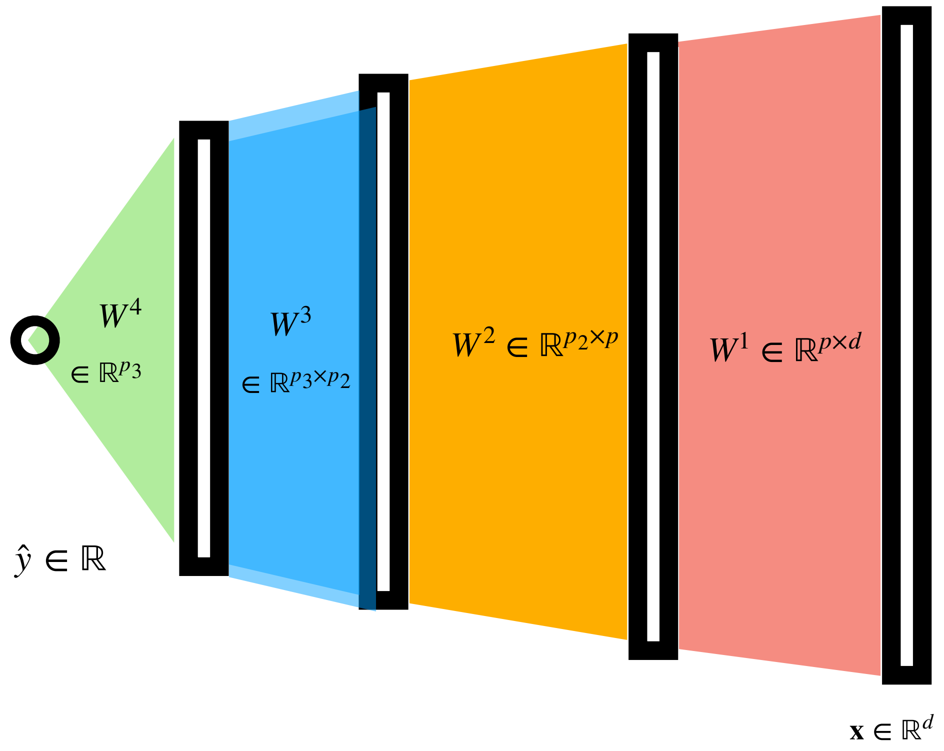

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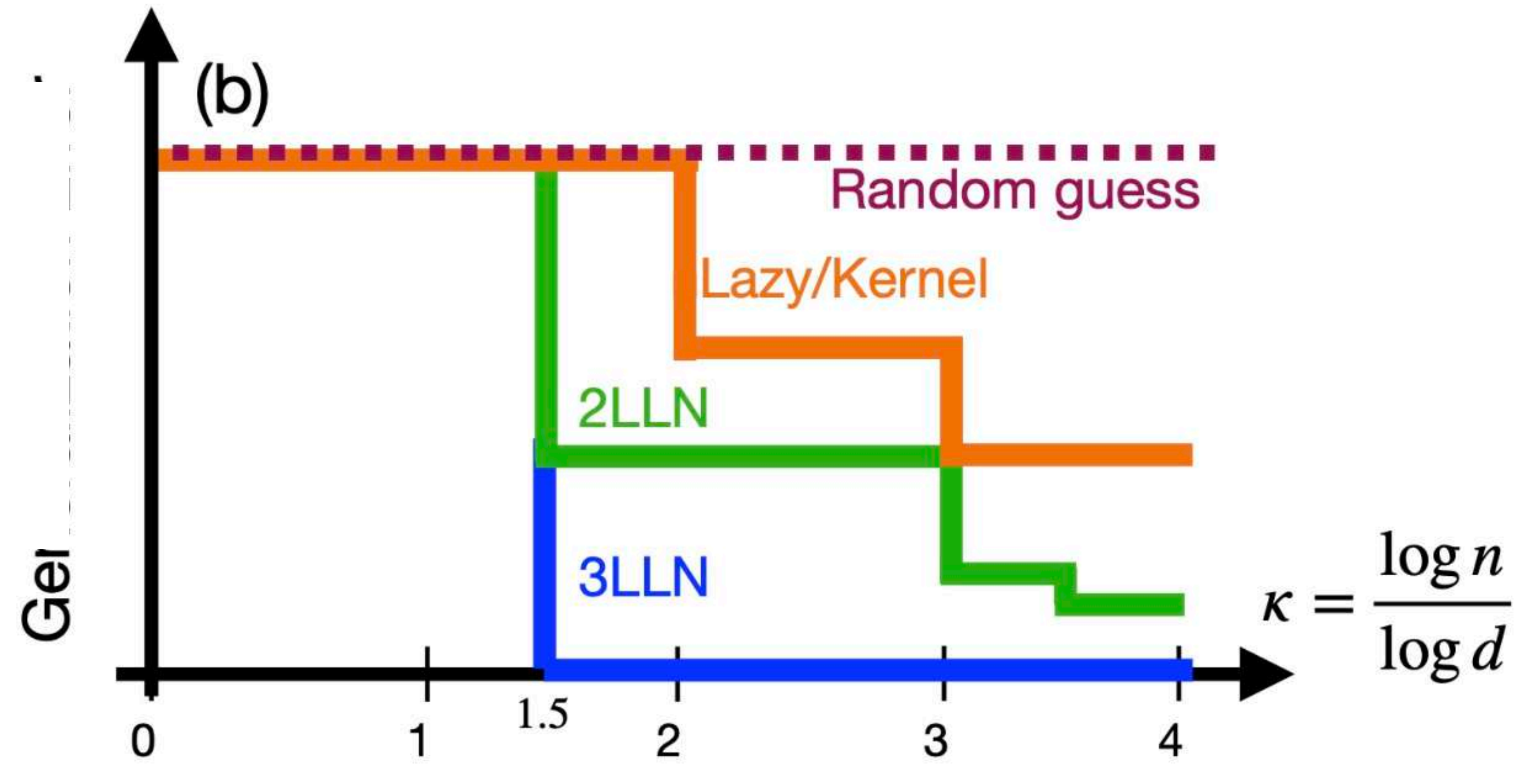
Theorem 2. *For $L \in \mathbb{N}$, let $f^\star(\mathbf{x})$ denote a target as in Eq. (6) with $r = 1$, and let δ', δ be arbitrary reals satisfying $0 < \delta < \delta' < 1$. Consider a model of the form $\hat{f}_\theta(\mathbf{x}) = \mathbf{w}_L^\top \sigma(W_{L-1}\sigma(W h^\star_{L-1}(\mathbf{x})))$ with $W \in \mathbb{R}^{p_{L-2}\times d^{\varepsilon_{L-2}}}$ having $p_{L-2} = \Theta(d^{k\varepsilon_{\ell-2}+\delta'})$ rows independently sampled as $\mathbf{w}_i \sim U(\mathbb{S}_{d^{\varepsilon_{L-2}}}(1))$. Under Ass. 1-3, after a single step of pre-conditioned SGD on W_{L-1} with batch-size $\Theta(d^{k\varepsilon_{\ell-2}+\delta})$, step-size $\Theta(\sqrt{p_{L-1}})$, the pre-activations $h_{L-1}(\mathbf{x}) := W_{L-1}\sigma(W h^\star_{L-1}(\mathbf{x}))$ satisfy, for a constant $c > 0$:*

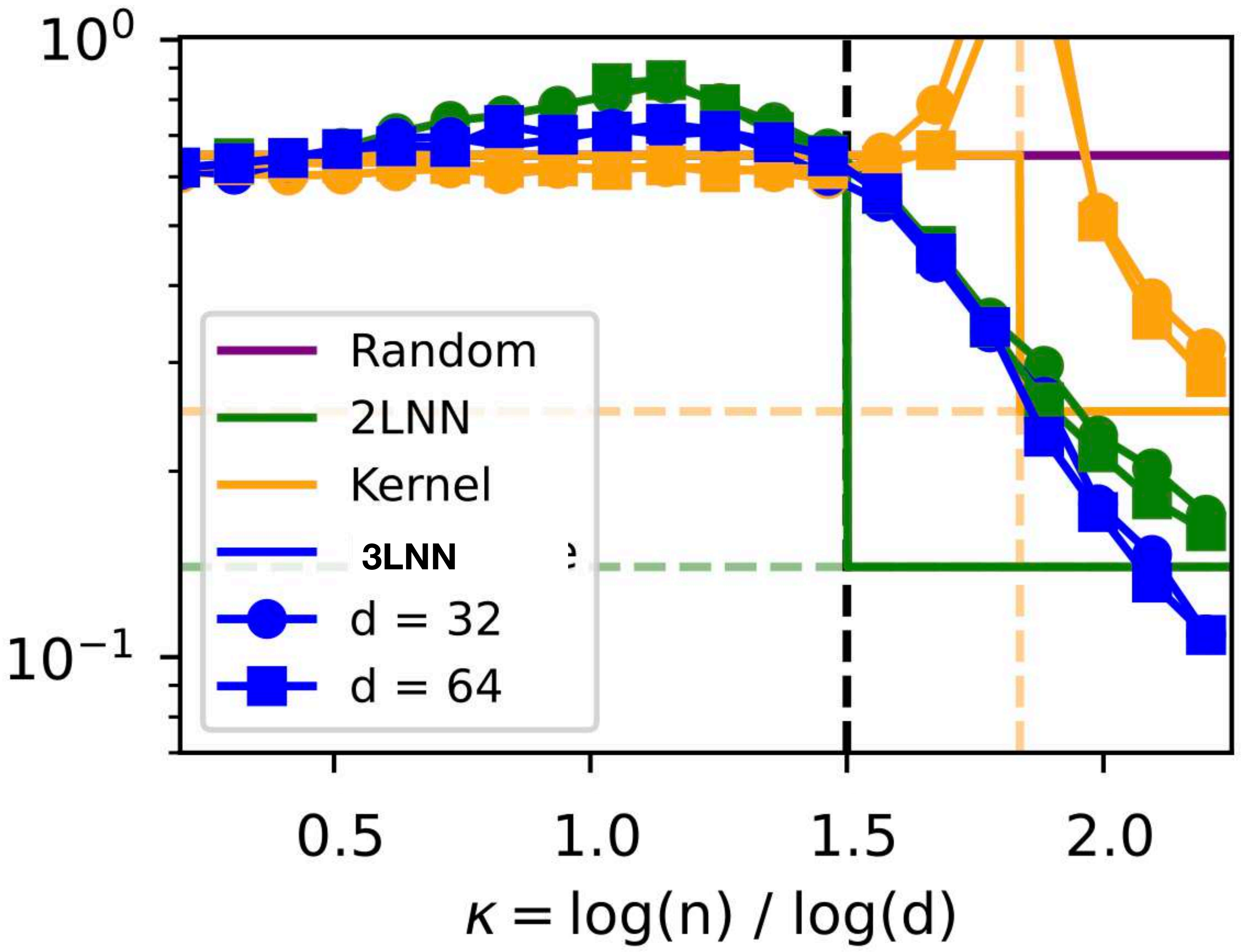
$$h_{L-1}(\mathbf{x}) = c\mathbf{w}_L h^\star_L(\mathbf{x}) + o_d(1), \tag{22}$$

Example

$$f^\star(\mathbf{x}) = \tanh\left(\frac{\mathbf{a}^{\star\top} P_3(W^\star \mathbf{x})}{\sqrt{d^{\varepsilon_1 = 1/2}}}\right)$$

$$P_3(x) = \mathrm{He}_2(x) + \mathrm{He}_3(x)$$





Overlaps in parameter and function space

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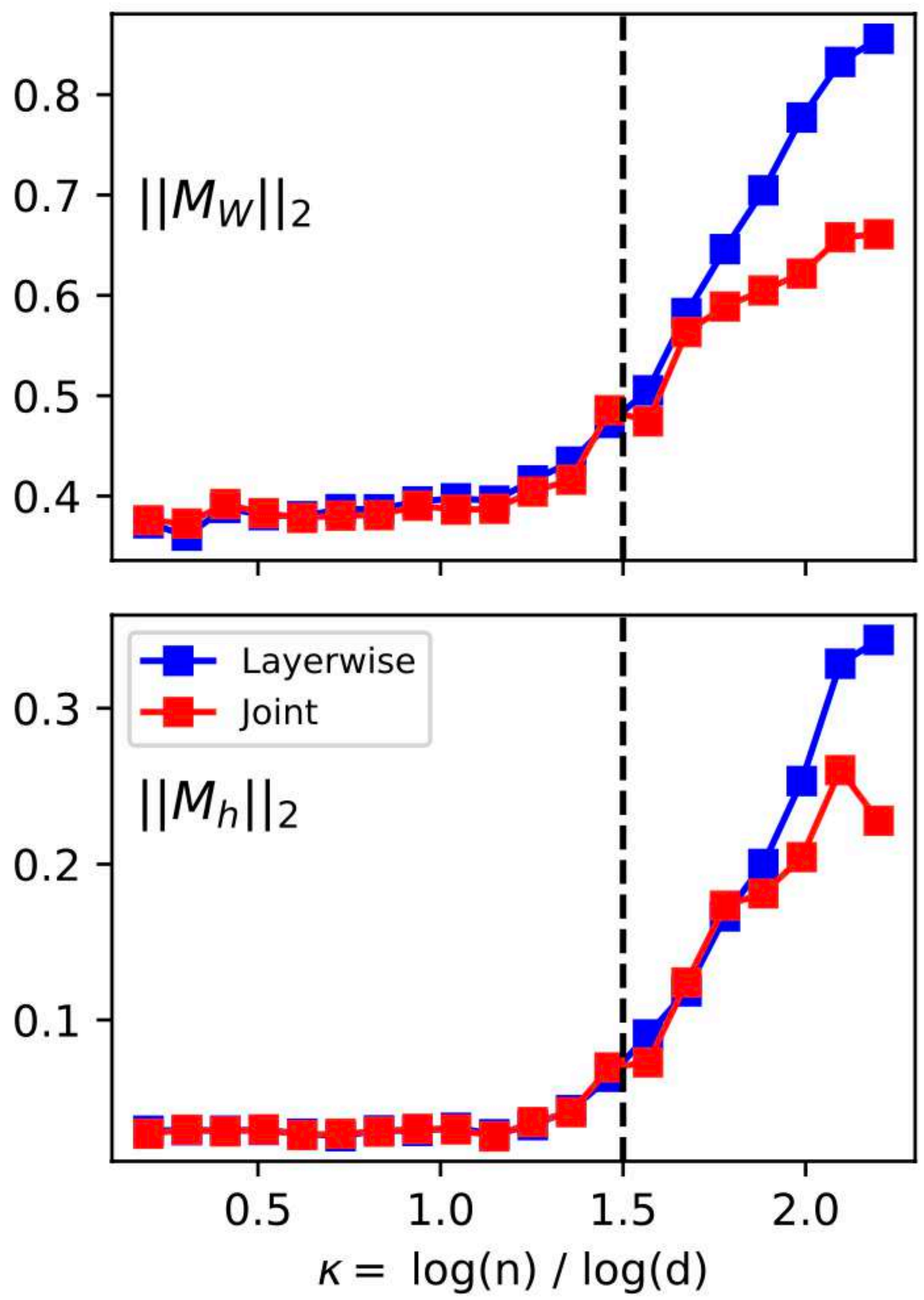
$$M_W = \frac{W_1 W^\star}{\|W_1\|_2},$$

$$M_h = \frac{\mathbb{E}[\mathbf{h}(\mathbf{z})\mathbf{h}^\star(\mathbf{z})]}{\sqrt{\mathbb{E}[\mathbf{h}(\mathbf{z})^2]}}.$$



V

Key ideas and proof sketches

Recursive Expansion of the Target

Recursive Expansion of the Target

$$f^\star(\mathbf{x}) = g^\star\left(\frac{\mathbf{a}^{\star\top} P_k\left(W^\star \mathbf{x}\right)}{\sqrt{d^\varepsilon}}\right), \mathbf{x} \in \mathbb{R}^d$$

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!!! $He_k(h^{\star}(\mathbf{x}))$ can contribute low-degree terms

Composition of Hermites

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$$h(\mathbf{x}) = \frac{\mathbf{1}}{\sqrt{\mathbf{d}^{\epsilon}}} \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{d}^{\epsilon}} \mathrm{He}_{\mathbf{k}}(\langle \mathbf{w}_{\mathbf{i}}^{\star}, \mathbf{x} \rangle)$$

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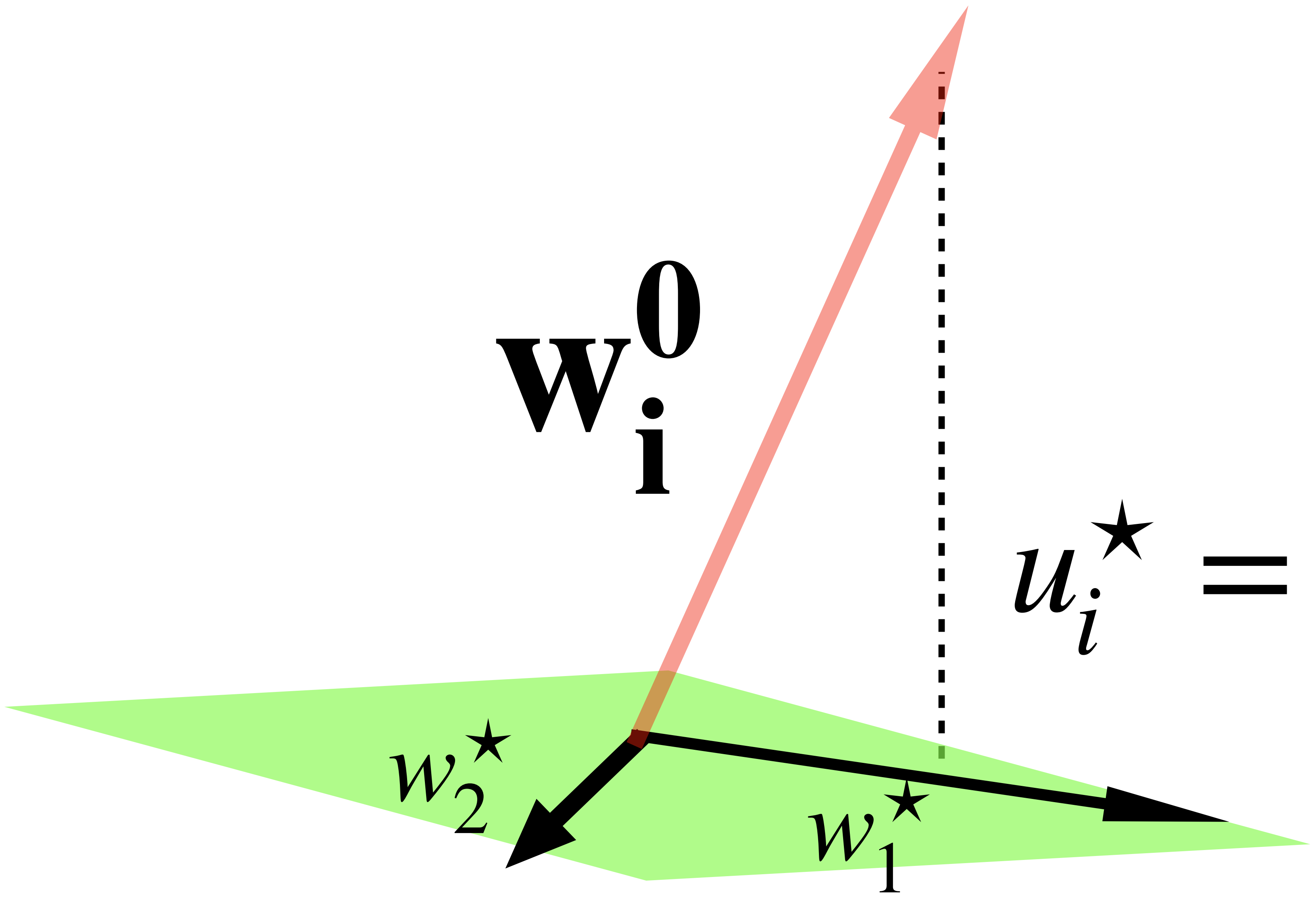

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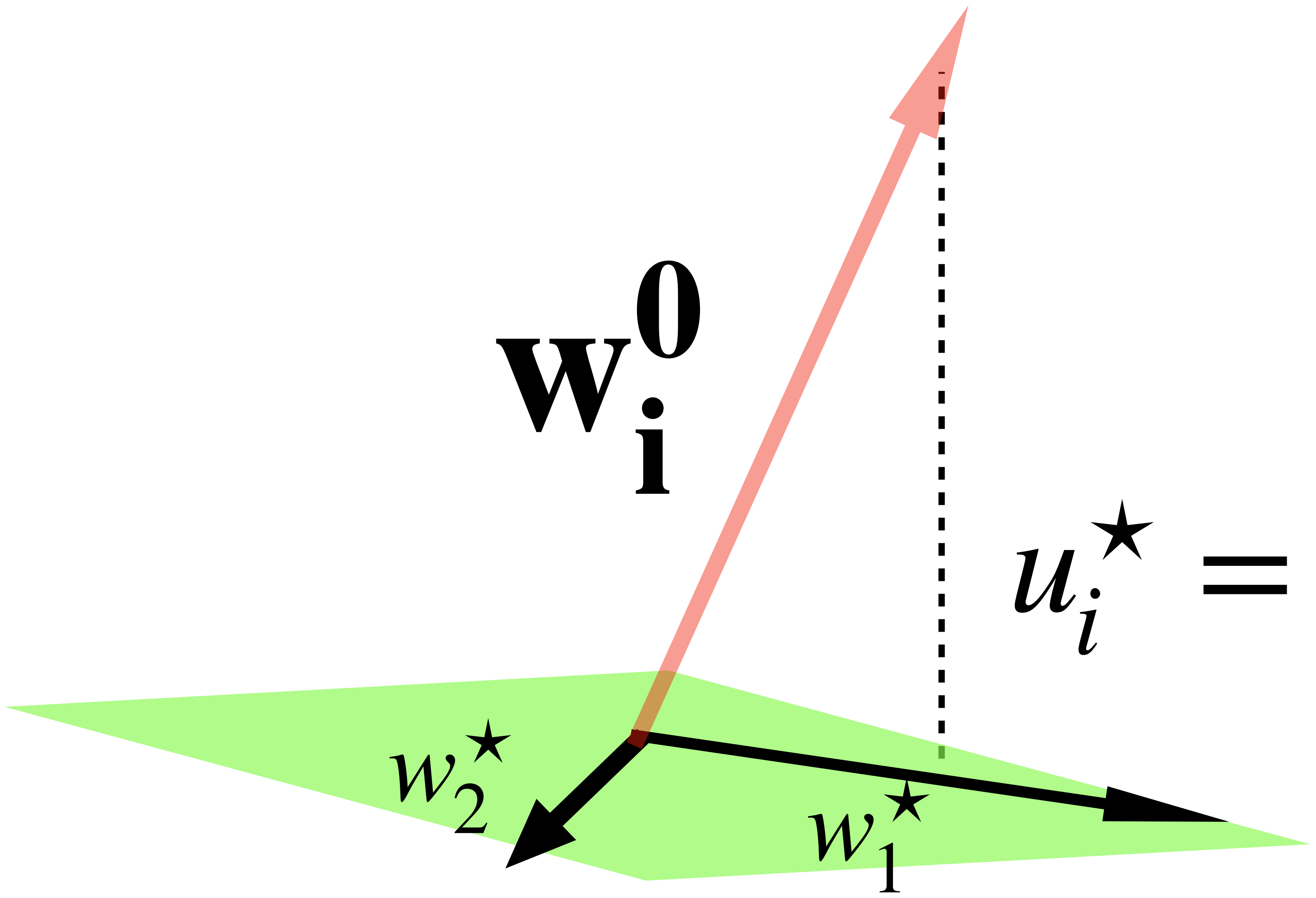

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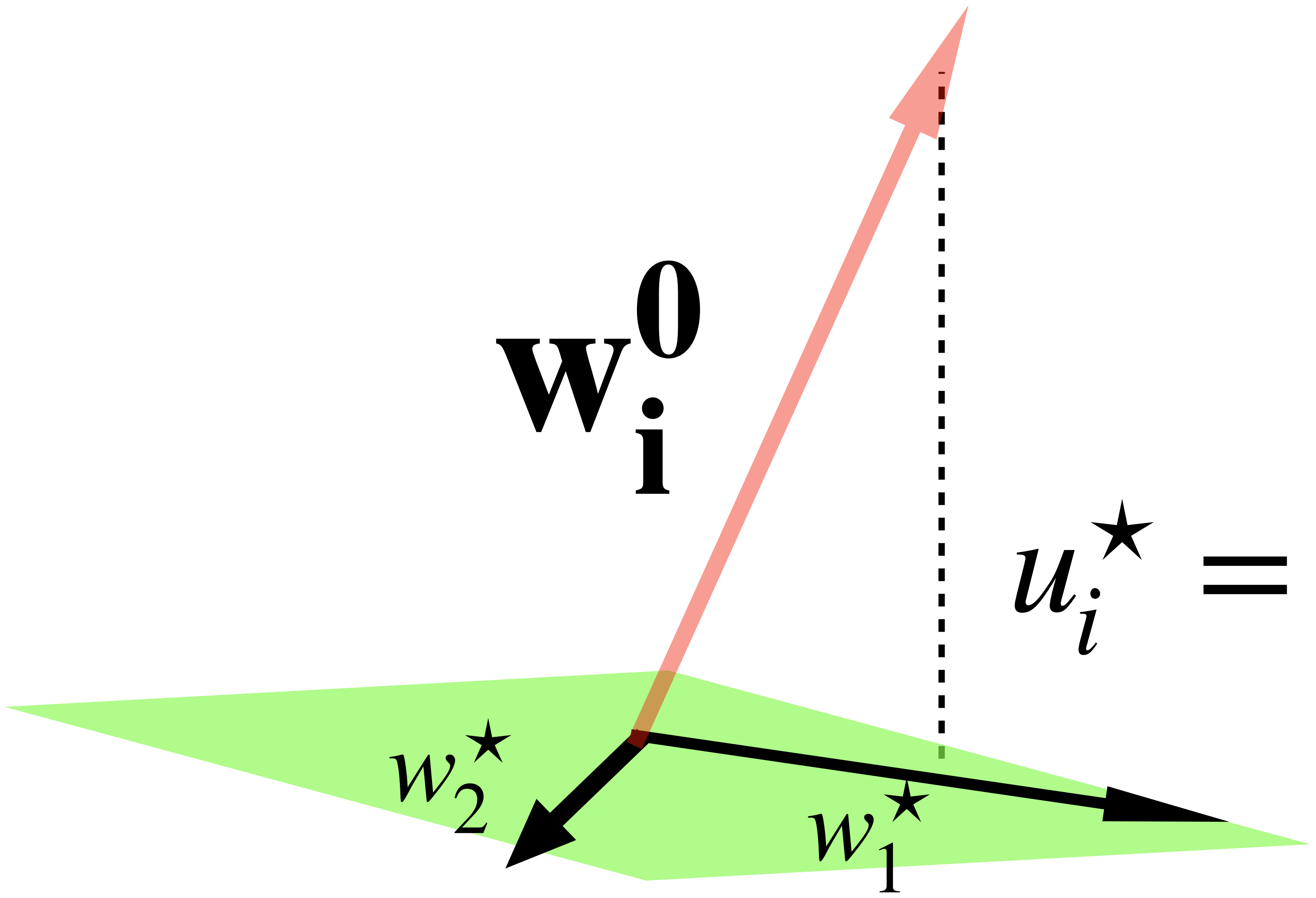

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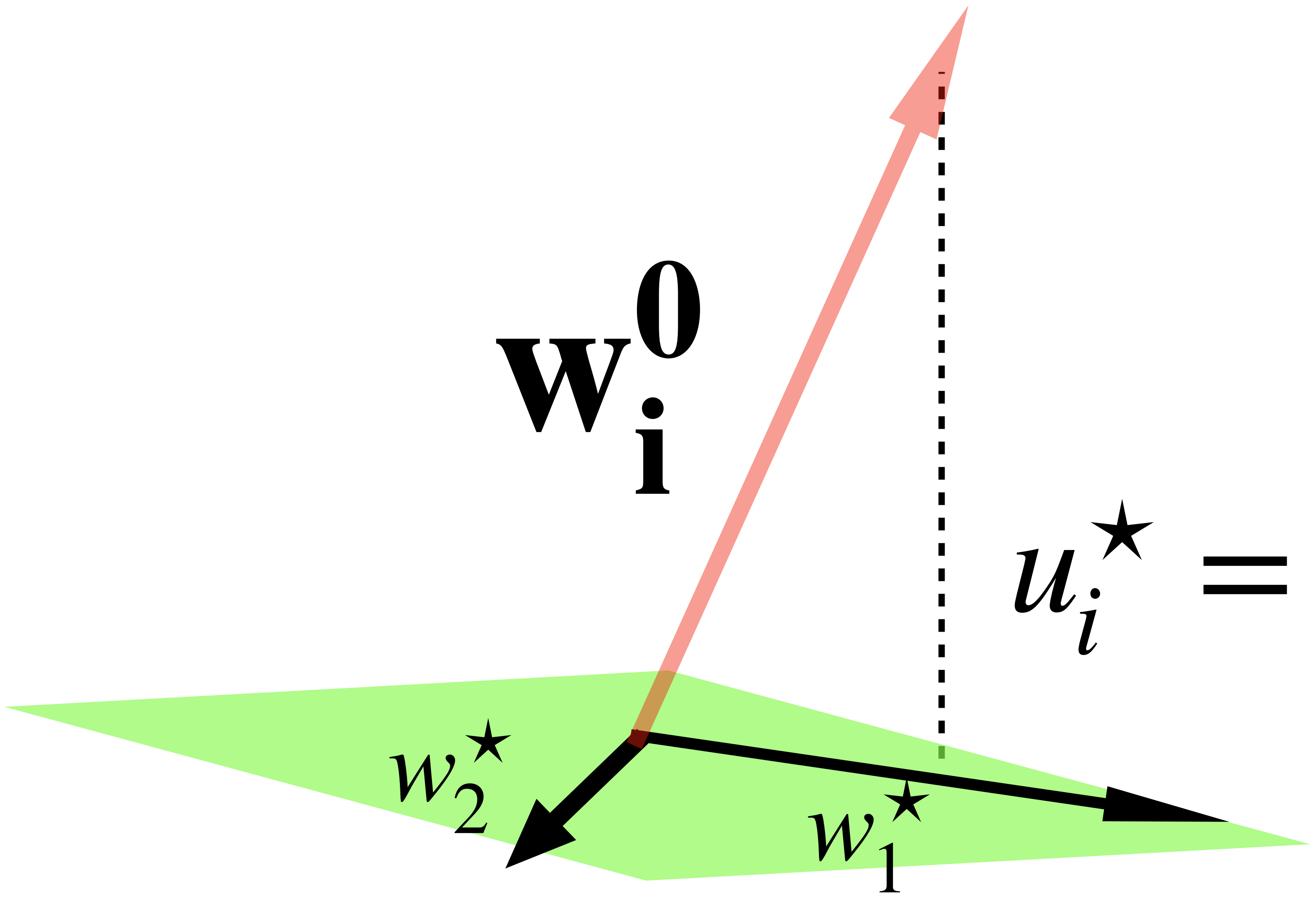

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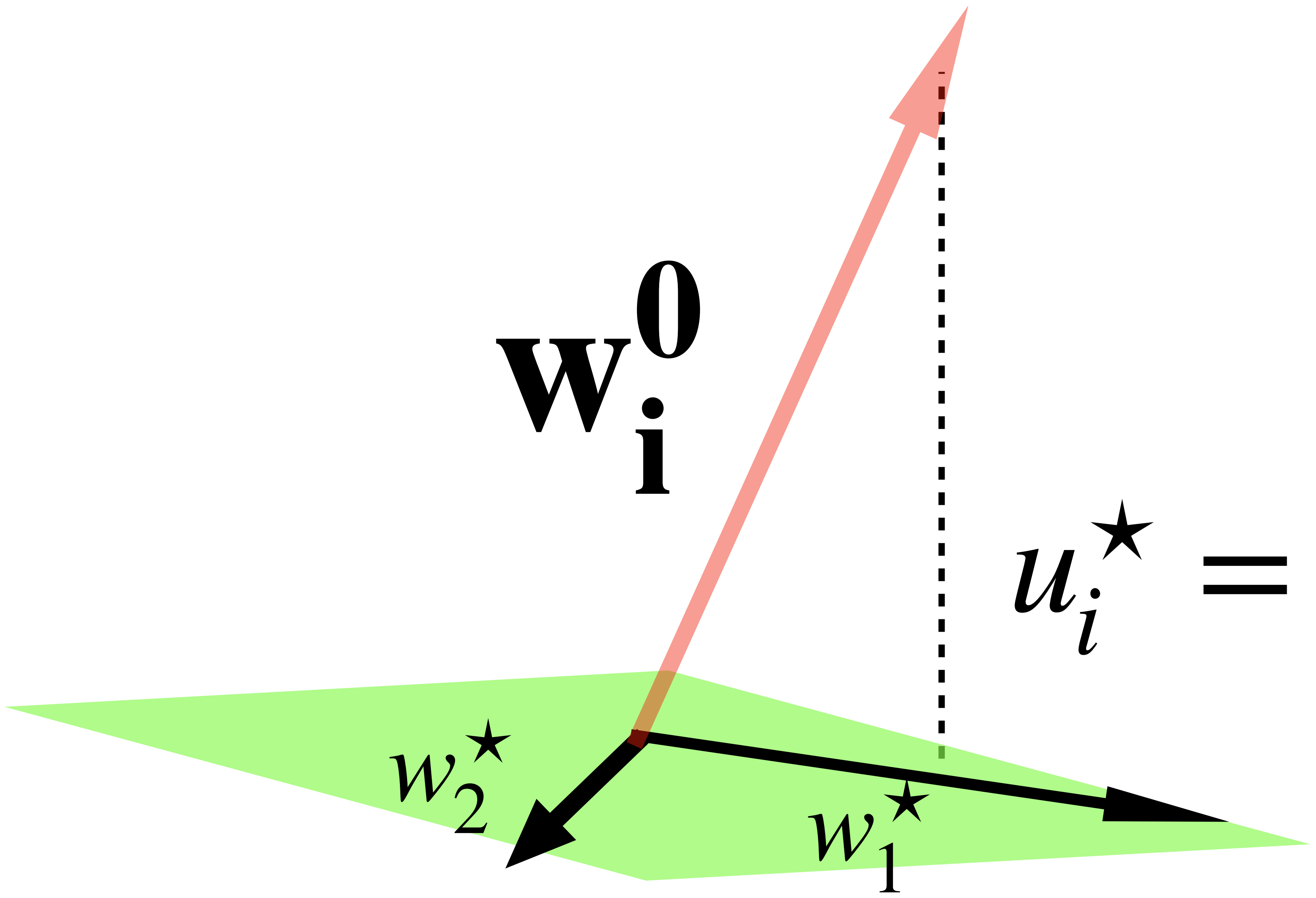

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higher-order terms supressed by vanishing specialization along $w_i^\star$ + vanishing step size.

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$$\mathbf{h}(\mathbf{x}) = W_2\sigma(\mathbf{W_1}\mathbf{x}) \in \mathbb{R}^{p_2}, X \in \mathbb{R}^{n\times d}$$

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Gradient updates $\approx$

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Gradient updates $\approx$ **Projections on the conjugate Kernel***

*Nichani et al. 2023

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Conjugate Kernel: $\sigma(\mathbf{W_1x})^\top \sigma(\mathbf{W_1X}^\top)$

Perturbed target: $\mathbf{f}^\star(\mathbf{X}) \odot \sigma'(\mathbf{h_i^t}(\mathbf{X}))$

*Nichani et al. 2023

Recovery by the second layer

$$\mathbf{h}(\mathbf{x}) = W_2\sigma(\mathbf{W_1x}) \in \mathbb{R}^{p_2}, X \in \mathbb{R}^{n\times d}$$

Gradient updates $\approx$ **Projections on the conjugate Kernel***

$$\Delta h^t_{i,2}(\mathbf{x}) \approx \langle \sigma(\mathbf{W_1x}), \mathbf{\Delta W_i} \rangle$$

$$\Delta W_2^i \propto w_i^3 \sigma(W_1 X^\top) f^\star(X) \odot \sigma'(\mathbf{h^t_i}(\mathbf{X})) \rightarrow \mathbf{\Delta h_i(x)} \propto \mathbf{w_i^3}\sigma(\mathbf{W_1x})^\top \sigma(\mathbf{W_1X}^\top)\mathbf{f}^\star(\mathbf{X}) \odot \sigma'(\mathbf{h^t_i}(\mathbf{X}))$$

Conjugate Kernel: $\sigma(\mathbf{W_1x})^\top \sigma(\mathbf{W_1X}^\top)$

Perturbed target: $\mathbf{f}^\star(\mathbf{X}) \odot \sigma'(\mathbf{h^t_i}(\mathbf{X}))$

$$\mathbf{\Delta h_i(x)} \propto \mathbf{w_i^3}\sigma(\mathbf{W_1x})^\top \sigma(\mathbf{W_1X}^\top)\mathbf{f}^\star(\mathbf{X})$$

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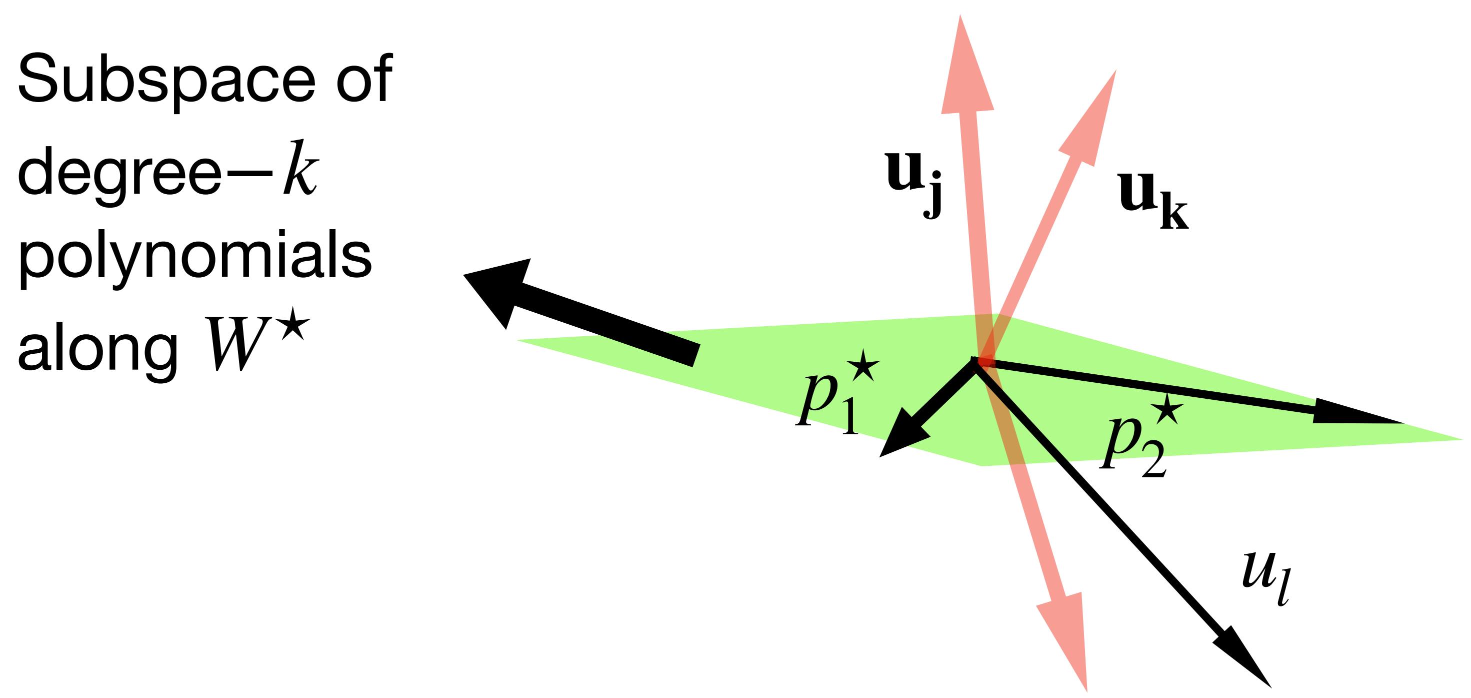

$K_{CK}(\mathbf{x}, \mathbf{x}')$ before training W_1

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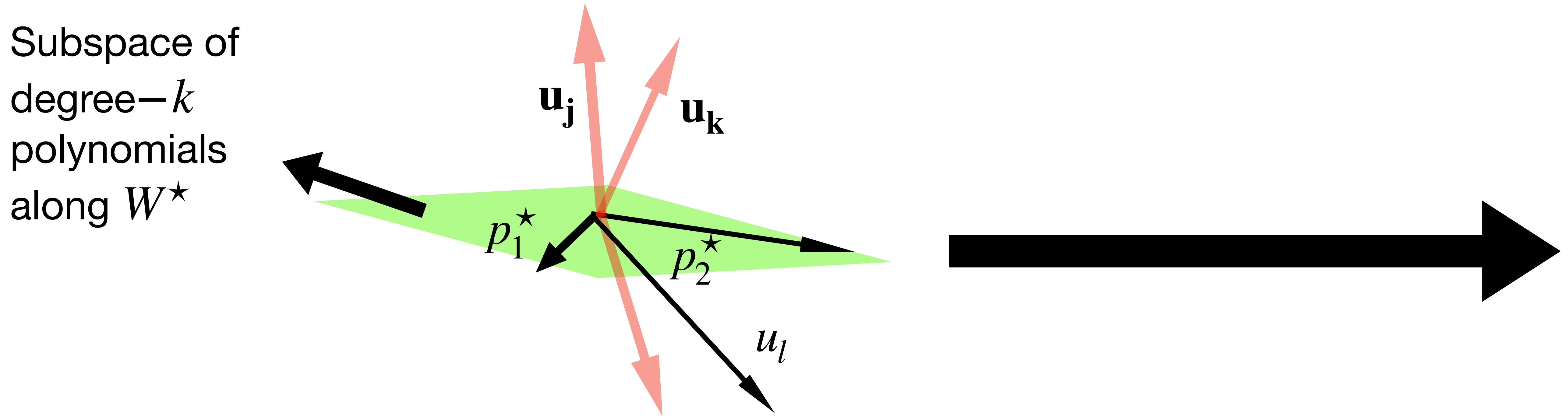

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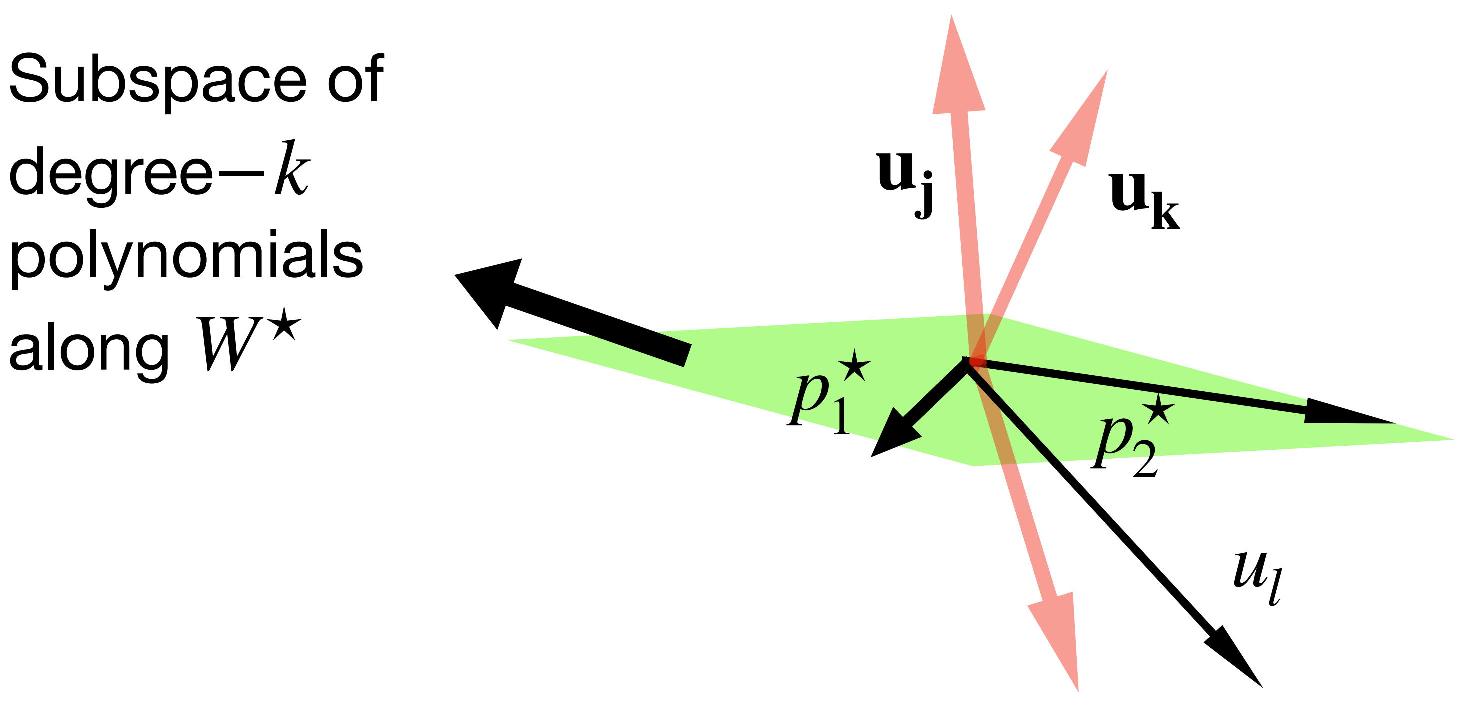

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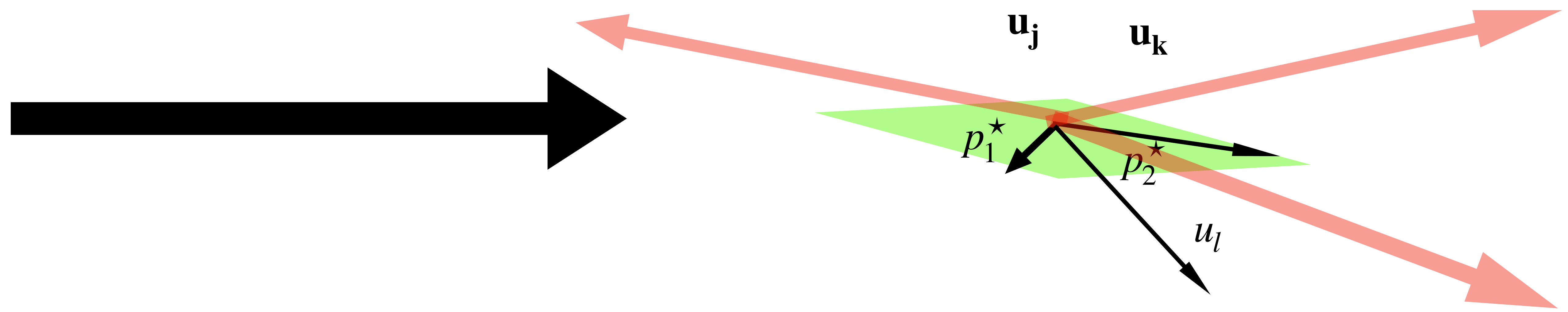

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Caveats:

- Conjugate Kernel ill-conditioned, $\lambda_k = \dfrac{1}{d^{\epsilon k}} \implies \mathcal{O}(d^{\epsilon k})$ steps for convergence $\implies \mathcal{O}(d^{2\epsilon k})$ sample complexity.
- Fix: Pre-conditioning: $W_2 = W_2 - \eta(\frac{1}{n}\sigma(W_1 X^\top)\sigma(W_1 X)^\top)^{-1} \nabla_{W_2}\mathscr{L}$

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$$K \approx \kappa_0^2 \mathbf{1}\mathbf{1}^T + \kappa_1^2 \frac{X_\star X_\star^T}{d} + \kappa_2^2 \frac{H_2(X_\star)H_2(X_\star)^T}{d^2} + \ldots + \kappa_k^2 \frac{H_k(X_\star)H_k(X_\star)^T}{d^k} + \kappa_{k+1}^2 \frac{H_{k+1}(X_\star)H_{k+1}(X_\star)^T}{d^{k+1}} + \ldots$$

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Generalization error of random features and kernel methods: hypercontractivity and kernel matrix concentration

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Caveat: With Gaussian inputs, some radial degree k polynomials require less than $O(d^k)$ samples.

Fitting the last layer

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- Reduction to Kernel on low-dimensional features.
- $K(\mathbf{x_1}, \mathbf{x_2}) = \mathbb{E}_{\mathbf{w}\sim\mathcal{N}(\mathbf{0},\mathbf{1})}[\sigma(\mathbf{cwh}^\star(\mathbf{x_1}) + \mathbf{b})\sigma(\mathbf{cwh}^\star(\mathbf{x_2}) + \mathbf{b})]$.
- $\hat{W}_3 \approx$ KRR$(K(\cdot), \mathbf{X}, \mathbf{y})$.
- Sample complexity/width now dimension-independent.

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Do we need narrowing of networks? No, consider:

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"approximately independent" $\mathcal{O}(d)$ features in $\mathcal{O}(d^2)$ space

Thanks to my collaborators!