Spin glasses, algorithms, and inference

Brice Huang (MIT → Stanford → Yale)

Statistical physics & machine learning: moving forward

Cargèse institute | August 14, 2025

Thanks to wonderful collaborators

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Nike Sun

Guy Bresler

Andrea Montanari

Huy Tuan Pham

Sidhanth Mohanty

Amit Rajaraman

David X. Wu

Lecture outline

1. Applications of planting in disordered models

2. A survey on the overlap gap property

Part I: applications of planting in disordered models

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- **Planted clique**: find a k-clique planted in $G(N, 1/2)$ (Jerrum 92, Ma Wu 13, Brennan Bresler 18+19+20, Lee Pernice Rajaraman Zadik 25)
- **Tensor PCA**: recover rank 1 spike planted in gaussian p-tensor (Montanari Richard 14, Hopkins Shi Steurer 15, Wein Alaoui Moore 19, Ben Arous Gheissari Jagannath 20, Ben Arous Gerbelot Piccolo 24)
- **Single/multi-index models**: recover $\boldsymbol{W}^*$ from $y_i = f(\boldsymbol{W}^* \boldsymbol{x}_i, \varepsilon)$ (Damian Lee Soltanolkotabi 22, Damian Pillaud-Vivien Lee Bruna 24, DLB 25, Troiani Dandi Defilippis Zdeborová Loureiro Krzakala 25)

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This talk: planted models are also useful as a **proof device** for studying "null" models **without planted signal**

Outline of part I: applications of planting in disordered models

The classic planting trick: planting a Gibbs sample

Ground state large deviations in spherical spin glasses

TAP planting: capacity of the Ising perceptron

Gibbs measures: prototypical examples

Sherrington–Kirkpatrick model: for $\boldsymbol{\sigma} \in \{\pm 1\}^N$, $\boldsymbol{W} \sim \mathrm{GOE}(N)$:

$$H(\boldsymbol{\sigma}) = \frac{1}{2}(\boldsymbol{W}\boldsymbol{\sigma}, \boldsymbol{\sigma}) \qquad \text{Gibbs measure: } \mu_{\beta H}(\boldsymbol{\sigma}) = \frac{1}{Z} e^{\beta H(\boldsymbol{\sigma})}$$

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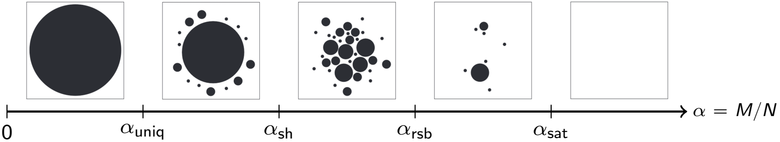
Applications:

- Spin glasses: deep connections to free energy
- Bayesian inference: model of posteriors; sampling applications

Gibbs measures: (predicted) geometric phase transitions



(Image from Krzakala Montanari Ricci-Tersenghi Semerjian Zdeborová 06)

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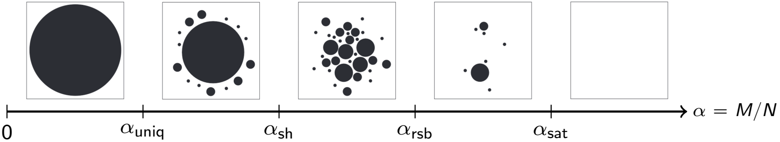

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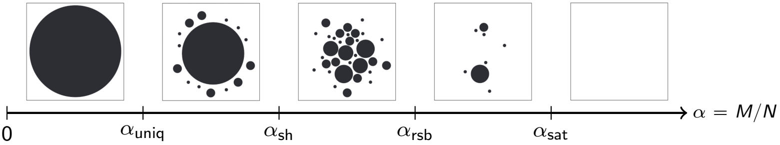

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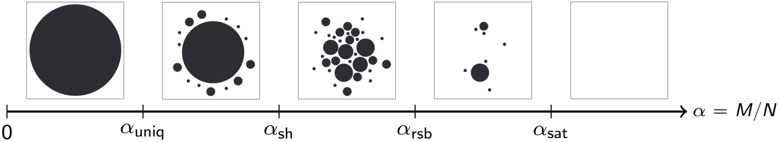

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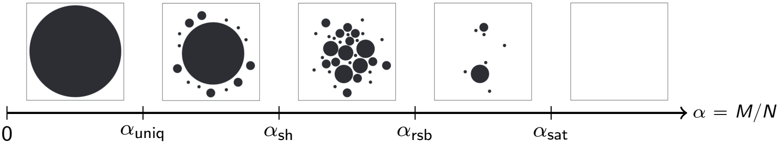

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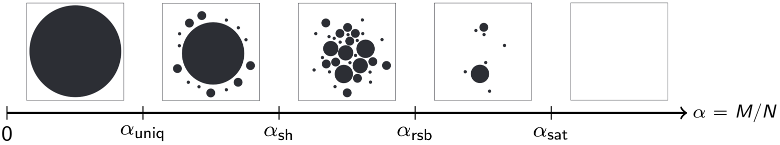

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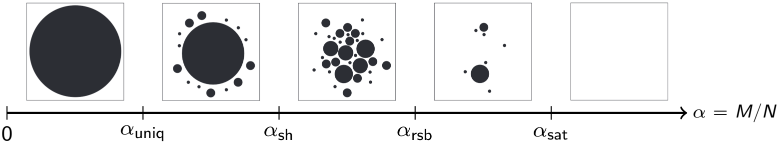

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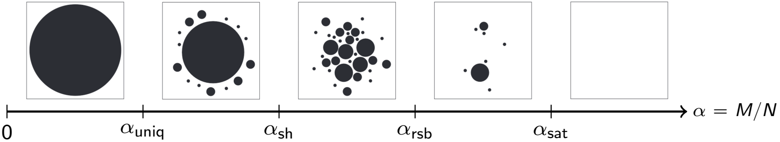

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Challenge: $\boldsymbol{\sigma} \sim \mu_H$ not very explicit and hard to work with.

Planted model (constraint satisfaction setting)

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$$\mathbb{P}_{\text{planted}}(\boldsymbol{E}) = 1 - o(1) \quad \longleftrightarrow \quad \mathbb{P}_{\text{null}}(\boldsymbol{E}) = 1 - o(1) \qquad \forall \text{ event } \boldsymbol{E}$$

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(hard)

Planted model:
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Relation between planted and null models

$\sigma \in \{\pm 1\}^N$

H

	×	×		×	×		×		×		
			×	×		×				×	
×					×						×
		×	×					×			×
							×	×		×	
×	×					×			×		

× indicates that σ satisfies H

Relation between planted and null models

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		×	×		×	×		×		×		
				×	×		×				×	
H	×					×						×
			×	×					×			×
								×	×		×	
	×	×					×			×		

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Null model: random row H, then
random × in that row

Relation between planted and null models

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H

	×	×		×	×		×		×		
			×	×		×				×	
×					×						×
		×	×					×			×
							×	×		×	
×	×					×			×		

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H

	×	×		×	×		×		×		
			×	×		×				×	
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Relation between planted and null models

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H

	×	×		×	×		×		×		
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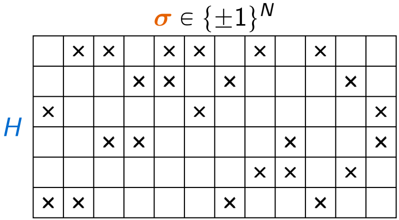

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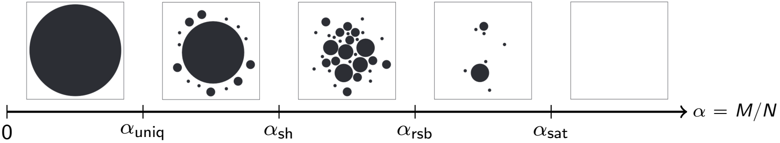
Holds for random k-NAE-SAT in **RS regime** $M/N < \alpha_{\text{rsb}}$

Application 1: shattering of random k-NAE-SAT



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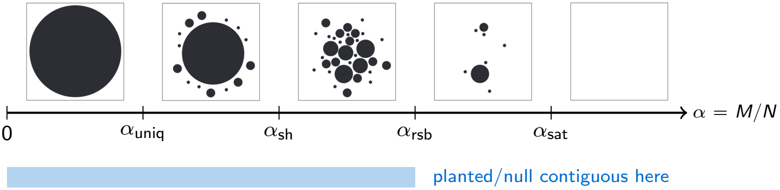

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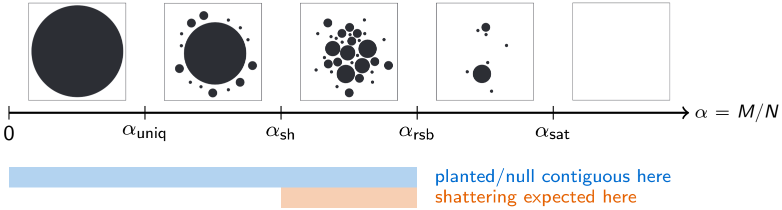

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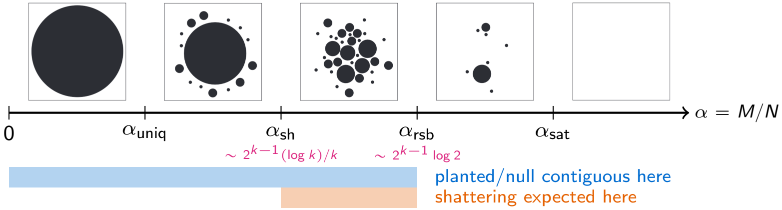

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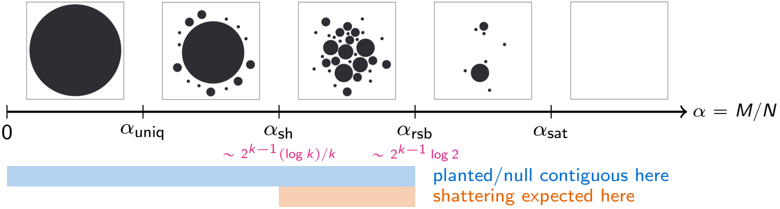

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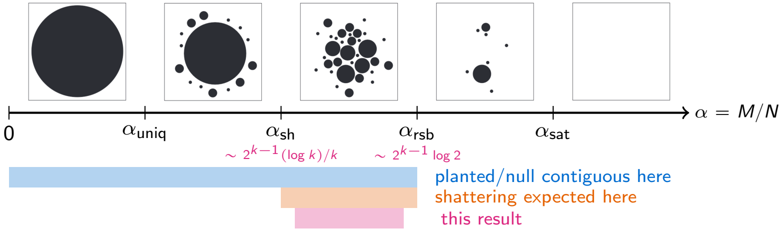

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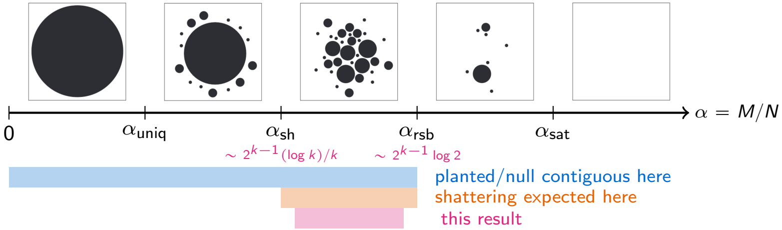

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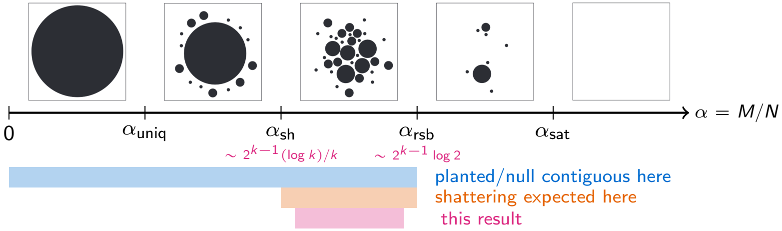
Application 1: shattering of random k-NAE-SAT



Theorem (Achlioptas Coja-Oghlan 08)

At constraint density $\alpha \in [(1 + o_k(1))\alpha_{\text{sh}}, (1 - o_k(1))\alpha_{\text{rsb}}]$, *whp over* ($k$*-NAE-SAT instance* H, *Gibbs sample* $\boldsymbol{\sigma}$)*:*

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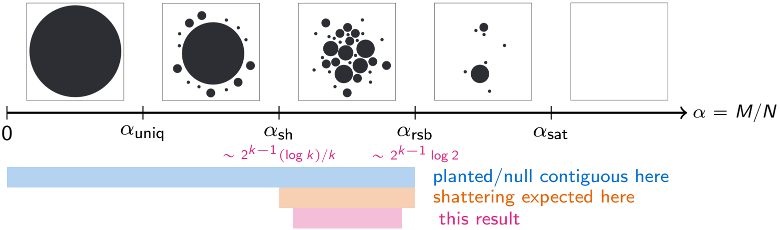

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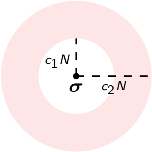

No sat assignments in ring around $\boldsymbol{\sigma}$ $\Rightarrow$ shattering

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$H \sim$ Law(random k-NAE-SAT | sat by $\boldsymbol{\sigma}$) has **explicit description**: clauses $\overset{IID}{\sim}$ Law(clause | sat by $\boldsymbol{\sigma}$).

Application 1: shattering of random k-NAE-SAT

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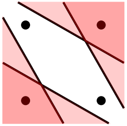
Application 2: symmetric Ising perceptron is frozen 1RSB



Symmetric Ising perceptron (Aubin Perkins Zdeborová 19):
intersection of discrete cube $\{\pm 1\}^N$ with IID symmetric slabs

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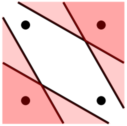

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Formally: for $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^M \overset{IID}{\sim} \mathcal{N}(0, I_N)$, $\kappa > 0$:

$$\boldsymbol{S} = \left\{ \boldsymbol{\sigma} \in \{\pm 1\}^N : |(\boldsymbol{g}^a, \boldsymbol{\sigma})| \leqslant \kappa\sqrt{N} \text{ for all } 1 \leqslant a \leqslant M \right\}$$

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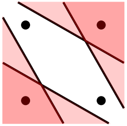

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Whp over $\boldsymbol{G}$ *and* $\boldsymbol{\sigma} \sim \mathrm{unif}(\boldsymbol{S})$, $\boldsymbol{\sigma}$ *has* **Hamming distance $\Omega(N)$** *to all other elements of* $\boldsymbol{S}$ **(frozen 1RSB)**.

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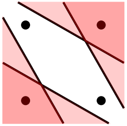

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Planted model: $\boldsymbol{\sigma} \sim \mathrm{unif}(\{\pm 1\}^N)$, then sample IID $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^M$ conditional on $|(\boldsymbol{g}^a, \boldsymbol{\sigma})| \leqslant \kappa\sqrt{N}$.

Application 3: shattering of pure spherical p-spin glass

Pure spherical p-spin model: for $g_{i_1,\ldots,i_p} \overset{IID}{\sim} \mathcal{N}(0,1)$,

$$H(\boldsymbol{\sigma}) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} g_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p}$$

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on domain $\boldsymbol{\sigma} \in S_N = \sqrt{N}\mathbb{S}^{N-1}$

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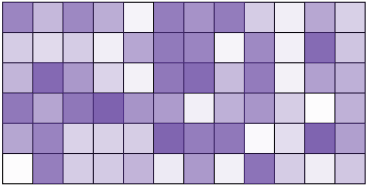
Equivalently: plant a spike

$$H(\boldsymbol{\rho}) = H_{\mathrm{null}}(\boldsymbol{\rho}) + N\beta R(\boldsymbol{\sigma}, \boldsymbol{\rho})^p \qquad R(\boldsymbol{\sigma}, \boldsymbol{\rho}) = \frac{(\boldsymbol{\sigma}, \boldsymbol{\rho})}{N}$$

Application 3: shattering of pure spherical p-spin glass

$\sigma \in S_N$

H



Heatmap indicates value of $e^{\beta H(\sigma)}$

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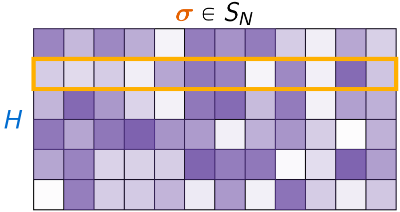

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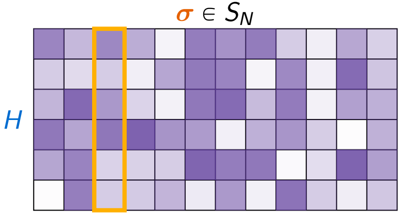
Null model: random row H, then col σ with prob $\propto e^{\beta H(\sigma)}$

Application 3: shattering of pure spherical p-spin glass



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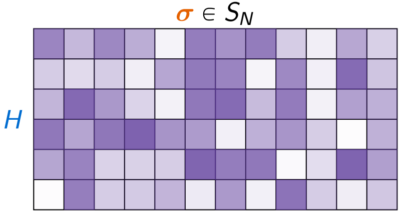

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(all cols symmetric, but rows aren't)

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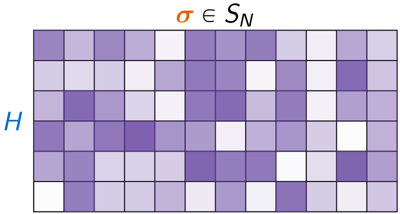

Heatmap indicates value of $e^{\beta H(\boldsymbol{\sigma})}$

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Planted model: random col $\boldsymbol{\sigma}$, then row H with prob $\propto e^{\beta H(\boldsymbol{\sigma})}$

(all cols symmetric, but rows aren't)

Planted model weights H by partition function ($\leftrightarrow$ row sum)

$$Z_\beta(H) = \int_{S_N} e^{\beta H(\boldsymbol{\sigma})} \, \mathrm{d}\boldsymbol{\sigma}$$

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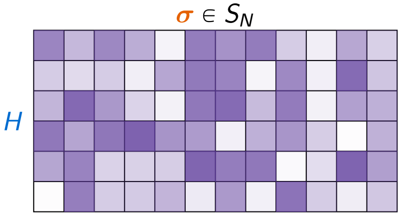

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That is, (similarly to before)

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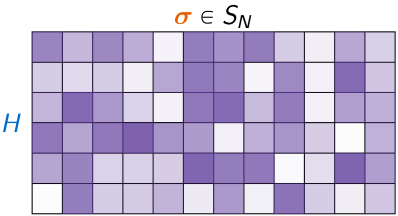

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Planted/null contiguous if this is $\Theta(1)$ whp. Holds for $\beta < \beta_{\text{rsb}}$.

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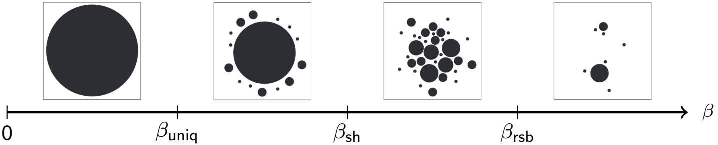

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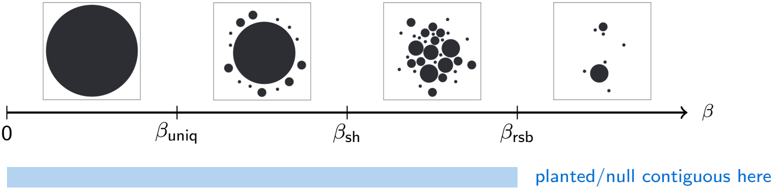

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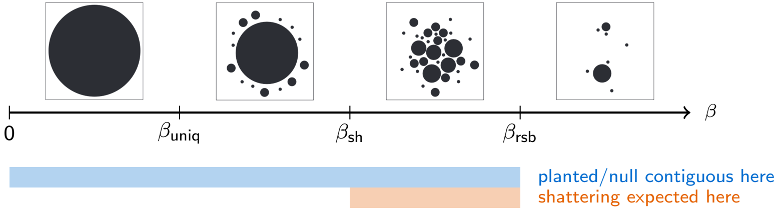

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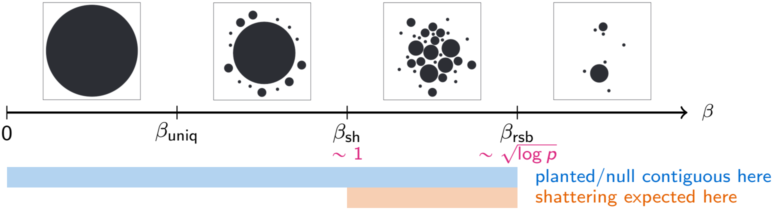

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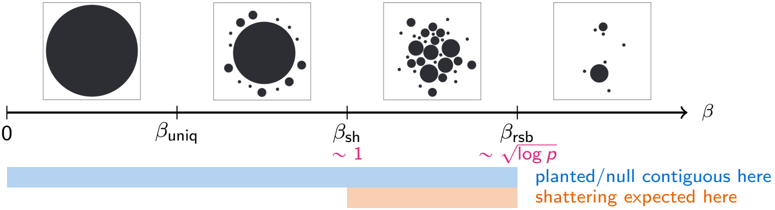
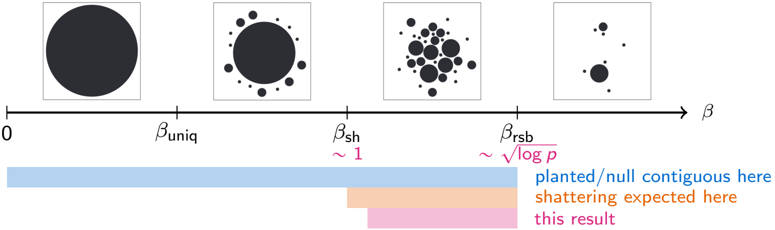


Theorem (El Alaoui Montanari Sellke 23)

For $\beta \in [\beta_{\text{sh}} \cdot O(1), \beta_{\text{rsb}})$,

Application 3: shattering of pure spherical p-spin glass



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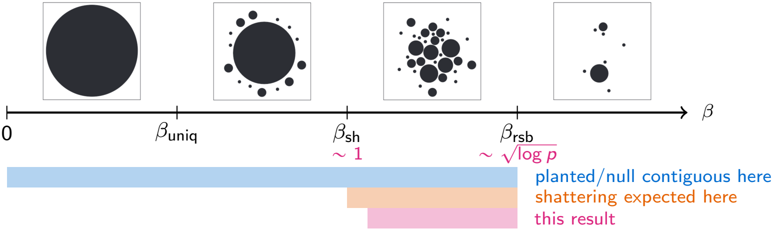

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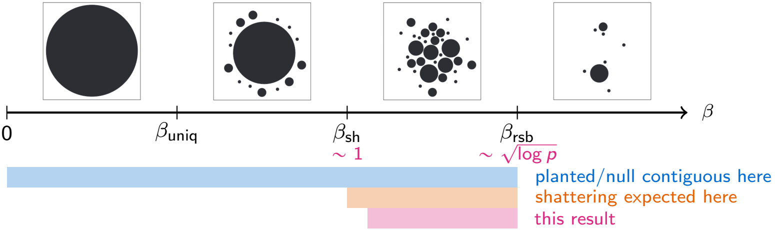

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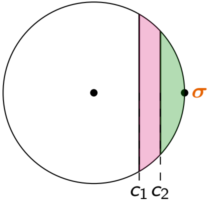

$\int e^{\beta H(\boldsymbol{\rho})}$ much larger in green region than red region $\Rightarrow$ shattering

Application 3: shattering of pure p-spin glasses

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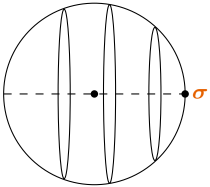

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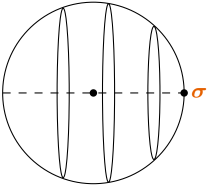

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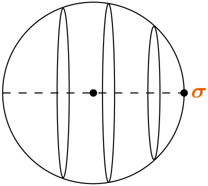

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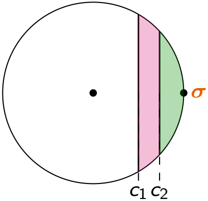
these show green region $\gg$ red region



Application 4: diffusion sampling from spin glasses

Main subroutine: mean ($\leftrightarrow$ score) estimation of following model

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El Alaoui Montanari Sellke 22+23, H Montanari Pham 24 use this to sample from Gibbs measure $\mu(\boldsymbol{\sigma}) \propto e^{H(\boldsymbol{\sigma})}$.

Other applications of planting

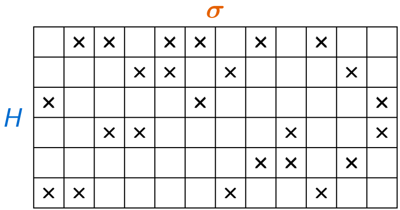
- Coja-Oghlan Krzakala Perkins Zdeborová 16
- Coja-Oghlan Efthymiou Jaafari Kang Kapetanopoulos 17
- Coja-Oghlan Kapetanopoulos Müller 18

} RS free energy of CSPs

- H Sellke 23: 2nd moment proof of RS free energy in spherical spin glasses

- Mossel Sly Sohn 24: sharp weak recovery threshold of sparse SBM

Classic planting requires centeredness + RS



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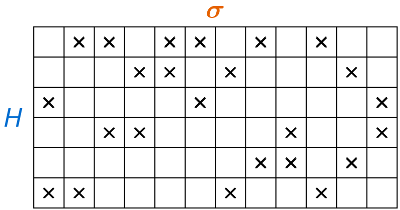

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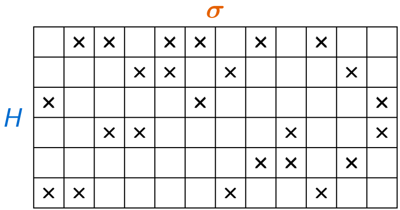

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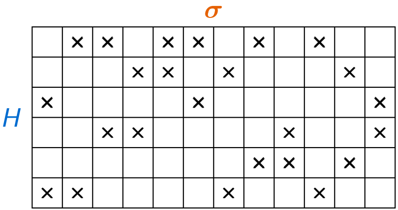

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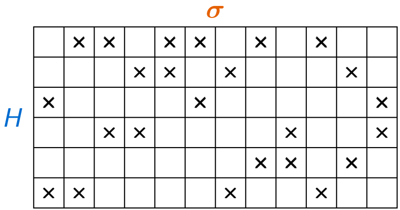

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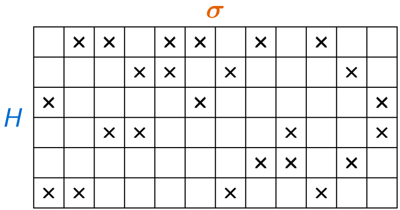

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Also need **free energy** $\approx$ **annealed free energy**: $\log Z = \log \mathbb{E}Z + O(1)$

We can plant things other than Gibbs samples!

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Rest of this half: two applications that each reduce to analyzing a planted model

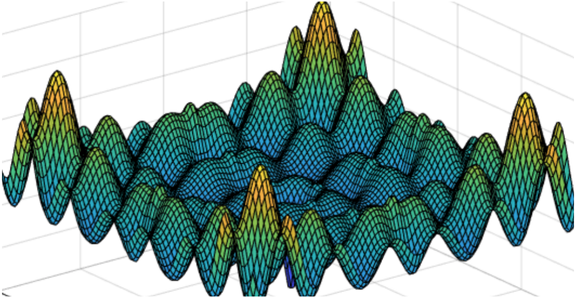
Outline of part I: applications of planting in disordered models

The classic planting trick: planting a Gibbs sample

Ground state large deviations in spherical spin glasses

TAP planting: capacity of the Ising perceptron

How complicated is a random landscape?



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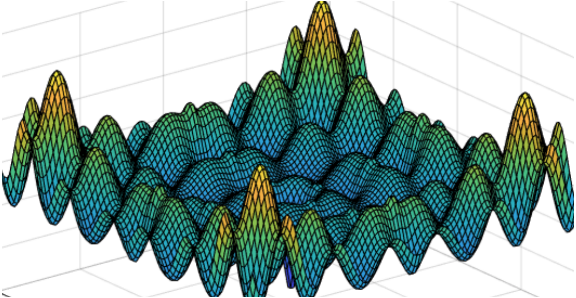

Fyodorov 04, Auffinger Ben Arous Černý 13: study via $\#$ critical points, using the **Kac–Rice formula** to calculate quantities like $\mathbb{E}[\#\,\text{crit pts}]$

Landscape complexity

Huge amount of work studying wide range of models:

• Subag 17, Ben Arous Subag Zeitouni 20, Belius Černý Nakajima Schmidt 22: spherical spin glasses

• Sagun Güney Ben Arous LeCun 14: neural networks

• Ben Arous Mei Montanari Nica 17: spiked tensor model

• Fyodorov 16, Ben Arous Fyodorov Khoruzhenko 21, Subag 23, Kivimae 24: non gradient vector fields

• Maillard Ben Arous Biroli 20: generalized linear models

• Fan Mei Montanari 21: TAP free energy in $\mathbb{Z}_2$-synchronization

• Ben Arous Bourgade McKenna 24: elastic manifold

• Kivimae 23, McKenna 24, H Sellke 25 : bipartite / multi-species spherical spin glasses

Ground state of pure spin glasses, via complexity

Auffinger Ben Arous Černý 13: crit pt complexity of pure p-spin model

$$H(\boldsymbol{\sigma}) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} g_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p}\,, \qquad g_{i_1,\ldots,i_p} \overset{IID}{\sim} \mathcal{N}(0,1)$$

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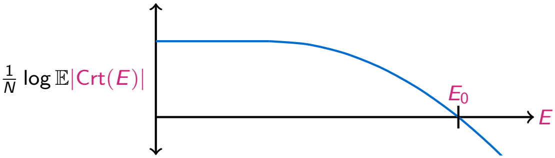

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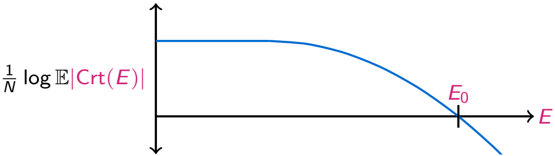

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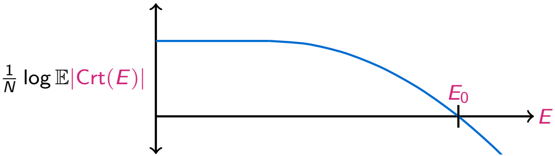

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This is sharp! E_0 matches ground state given by **Parisi formula**.

Related works on landscape complexity and ground state

- Subag 17: $\mathrm{GS}_N \geqslant E_0$ via 2nd moment analysis of $|\mathrm{Crt}(E)|$.

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Upper-tail large deviations of ground state

For $E > E_0$, what is the large deviation rate

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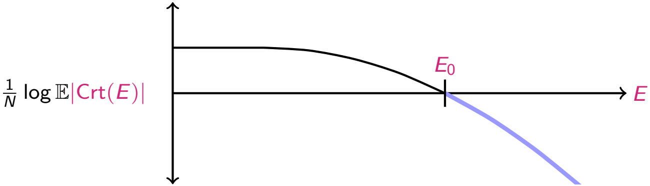
Implies upper bound on $\Phi(E)$:



Upper-tail large deviations of ground state

For $E > E_0$, what is the large deviation rate

$$\Phi(E) = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}(\mathrm{GS}(H) \geqslant E) \quad \text{where} \quad \mathrm{GS}(H) = \max_{\boldsymbol{\sigma}\in S_N} \frac{H(\boldsymbol{\sigma})}{N} \,?$$

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Theorem (Subag 17; Ben Arous Subag Zeitoni 20; H Sellke 23)

Yes. In fact, in all pure p-spin models,

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(H Sellke 23: also in a maximal regime of mixed p-spin models — later)

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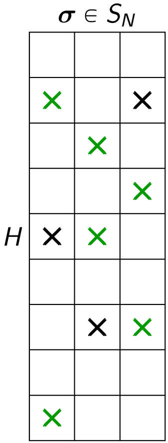

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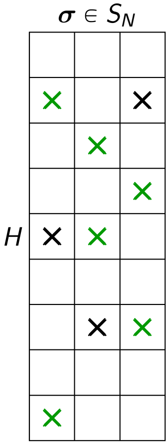

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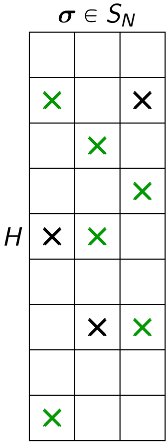

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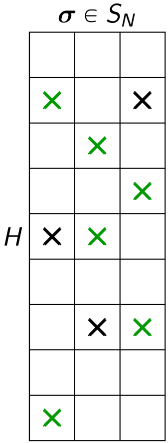

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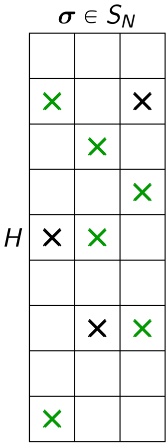

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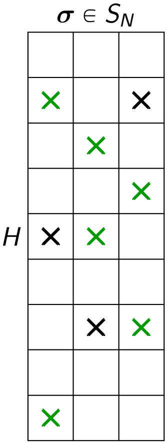

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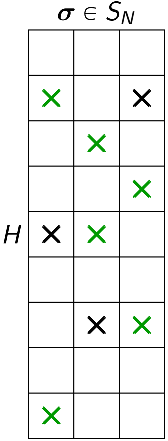
want to show: this is $1 - o(1)$

Re-interpretation: critical point planted model



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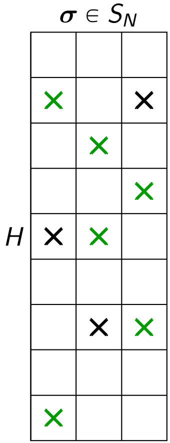

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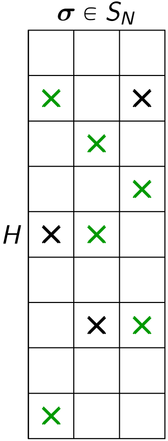

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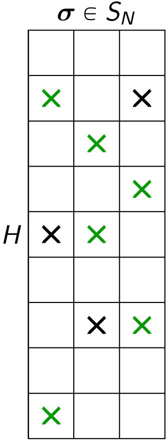

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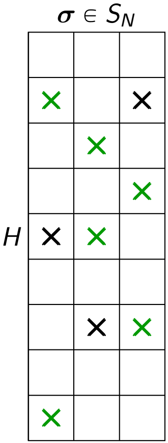

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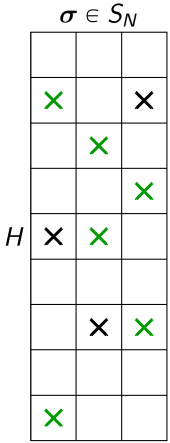

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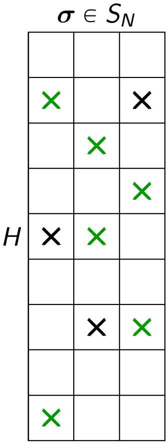

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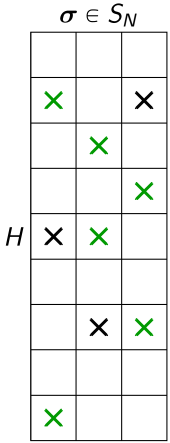

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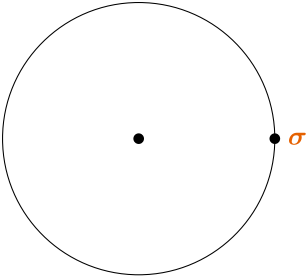

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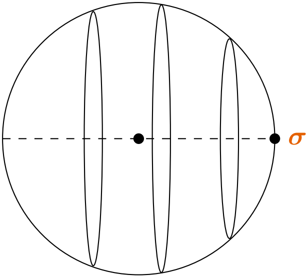

On each orthogonal band, max of H bounded by **Guerra's interpolation**

Beyond pure models

Q: does $\mathbb{P}(\mathrm{GS}(H) \geqslant E) = (1 - o(1))\mathbb{E}|\mathrm{Crt}(E)|$ in **mixed** p**-spin model**?

$$H(\boldsymbol{\sigma}) = \sum_{p \geqslant 2} \frac{\gamma_p}{N^{(p-1)/2}} \sum_{i_1, \ldots, i_p = 1}^{N} g_{i_1, \ldots, i_p} \sigma_{i_1} \cdots \sigma_{i_p}$$

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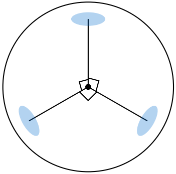

As $\beta \to \infty$, Gibbs measure

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concentrates on **orthogonal** spherical caps whp.

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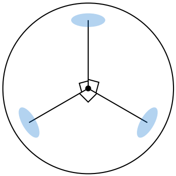

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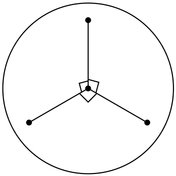
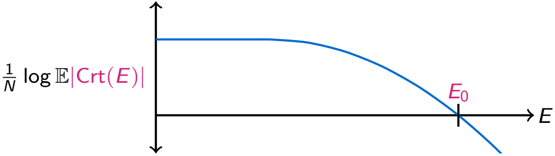


Equivalently: crit pts of H with value $\approx \mathrm{GS}_N$ are whp orthogonal.

(That is, they **do not cluster**)

Complexity-based proof of ground state energy

Auffinger Ben Arous Černý 13 + Subag 17: in **pure p-spin models**, complexity-based proof of $\mathrm{GS}_N \xrightarrow{p} E_0$. Independent of Parisi formula.



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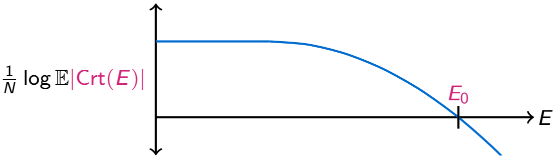

Ben Arous Subag Zeitouni 20: similarly $\mathrm{GS}_N \xrightarrow{p} E_0$ in **some regime of mixed p-spin models**

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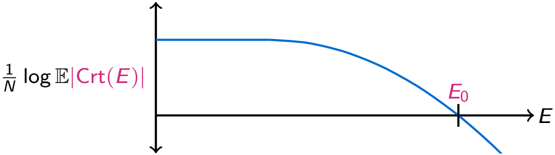

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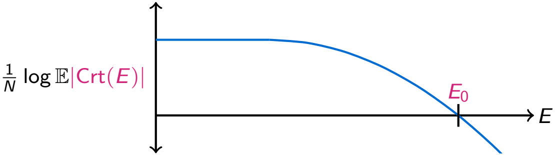

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(uses Guerra interpolation, but avoids more difficult Parisi formula LB)

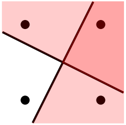
Outline of part I: applications of planting in disordered models

The classic planting trick: planting a Gibbs sample

Ground state large deviations in spherical spin glasses

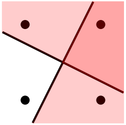
TAP planting: capacity of the Ising perceptron

The perceptron model



Intersection of discrete cube $\{\pm 1\}^N$
with M i.i.d. random half-spaces

The perceptron model



Intersection of discrete cube $\{\pm 1\}^N$ with M i.i.d. random half-spaces

Formally: for $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^M \sim \mathcal{N}(0, I_N)$,

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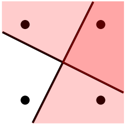

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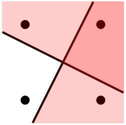

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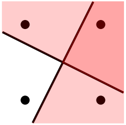

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$\leftrightarrow$ memorization capacity of a neural network (Gardner 87)

Main result

Conjecture (Krauth Mézard 89)

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(next slide)

Both results hold for more general model with **margin** $\kappa \in \mathbb{R}$:

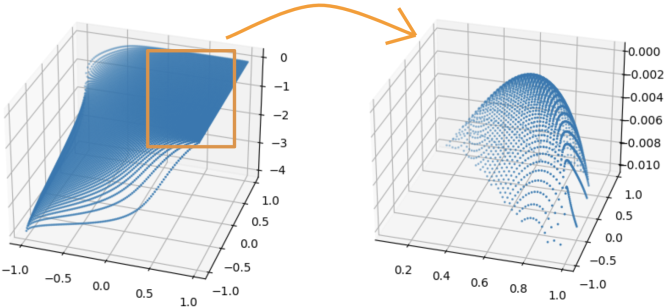
$$S = \left\{ \boldsymbol{x} \in \{\pm 1\}^N : (\boldsymbol{g}^a, \boldsymbol{x}) \geqslant \kappa\sqrt{N}, \quad \forall 1 \leqslant a \leqslant M \right\}$$

for analogous threshold $\alpha_{\mathsf{KM}}(\kappa)$, under further numerical conditions depending on κ.

The function in our numerical condition

$\mathscr{S}_\star(1,0) = 0$ local max, conjecturally unique global max

Plot of $\mathscr{S}_\star$ (domain $\mathbb{R}^2$ reparametrized to $[-1,1]^2$):



$x, y \in [-1, 1]$ $\mathscr{S}_* \geq -0.01$

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Review: 1st/2nd moment method

- $\mathbb{E}|\boldsymbol{S}(N\boldsymbol{\alpha})| \ll 1 \Rightarrow$ no solution at constraint density $\boldsymbol{\alpha}$ (whp)
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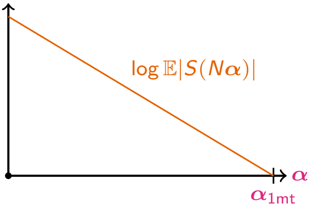

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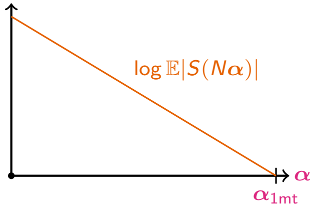
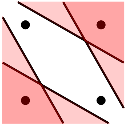


- (Hope to) show $\mathbb{E}[|\boldsymbol{S}(N\boldsymbol{\alpha}_{1\text{mt}})|^2] \asymp (\mathbb{E}|\boldsymbol{S}(N\boldsymbol{\alpha}_{1\text{mt}})|)^2 = 1$.

If so, $\boldsymbol{\alpha}_\star = \boldsymbol{\alpha}_{1\text{mt}}$.

1st/2nd moment method: a success story

Symmetric Ising perceptron: constraints $|(\boldsymbol{g}^a, \boldsymbol{x})| \leqslant \boldsymbol{\kappa}\sqrt{N}$
(Aubin Perkins Zdeborová 19, Perkins Xu 21, Abbe Li Sly 22, ...)



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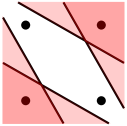

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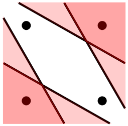

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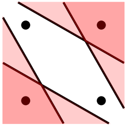

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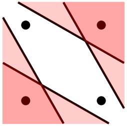

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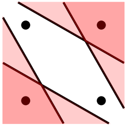

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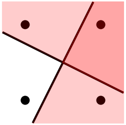
Can similarly calculate $\mathbb{E}[|\boldsymbol{S}(M)|^2]$, verify $\mathbb{E}[|\boldsymbol{S}(M)|^2] \asymp (\mathbb{E}|\boldsymbol{S}(M)|)^2$.

Direct moment method fails in asymmetric model

In our model, $\boldsymbol{S}(\alpha N) = \{\boldsymbol{x} \in \{\pm 1\}^N : \ (\boldsymbol{g}^a, \boldsymbol{x}) \geqslant 0 \quad \forall 1 \leqslant a \leqslant \alpha N\}$



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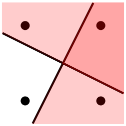

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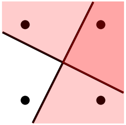

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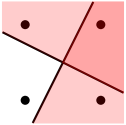

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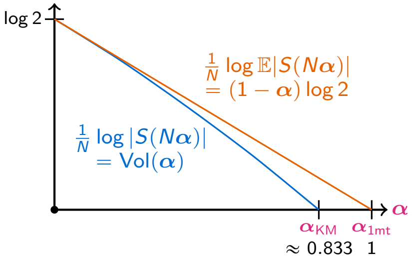
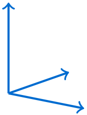
Our approach: pass to a contiguous planted model in which 1st/2nd moment method locates capacity.
Next few slides motivate choice of planted model.

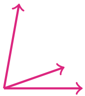
What goes wrong? A large deviations perspective



$\mathbb{E}|S(N\alpha)|$ dominated by events where the g^a are **atypically correlated**



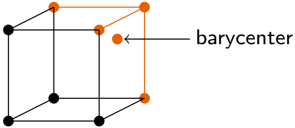
Typically: g^a orthogonal



Atypically: g^a correlated, which inflates # solutions

Remedy: conditional moment method

Key intuition of Ding Sun 18, Bolthausen 19: 1st mt failure caused by large deviation events in **barycenter** of solution set ***S***



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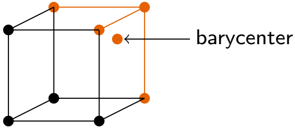

That is, $\mathbb{E}(|\boldsymbol{S}|) \gg (\text{typical } |\boldsymbol{S}|)$ but we expect, for **typical** realization of barycenter:

$$(\text{typical } |\boldsymbol{S}|) \quad \asymp \quad \mathbb{E}(|\boldsymbol{S}| \,|\, \text{barycenter})$$

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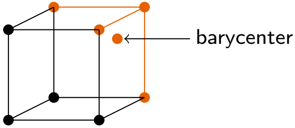

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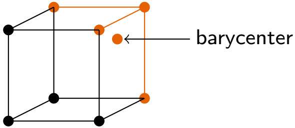

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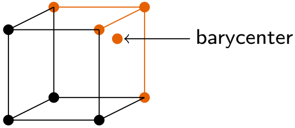

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Will implement by **planting** a certain **heuristic proxy** of barycenter

Heuristic description of barycenter

TAP equation (Thouless Anderson Palmer 77): nonlinear system in

- $\boldsymbol{G} \in \mathbb{R}^{M \times N}$ matrix with rows $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^M$
- $\boldsymbol{m} \in \mathbb{R}^N$ barycenter of $\boldsymbol{S}$
- $\boldsymbol{n} \in \mathbb{R}^M$ average slacks of constraints: $n_a = \mathrm{avg}_{\boldsymbol{x} \in \boldsymbol{S}} \{ (\boldsymbol{g}^a, \boldsymbol{x})/\sqrt{N} \}$

Heuristic description of barycenter

TAP equation (Thouless Anderson Palmer 77): nonlinear system in

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For explicit nonlinearities $\dot{F}, \widehat{F} : \mathbb{R} \to \mathbb{R}$, constants b, d:

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Physics prediction: whp over $\boldsymbol{G}$, this has a unique solution $(\boldsymbol{m}, \boldsymbol{n})$ (which has the physical meaning above)

Key idea: TAP planted model

Null model:

- $\boldsymbol{G} \in \mathbb{R}^{M \times N} \sim$ IID $\mathcal{N}(0,1)$ entries
- $(\boldsymbol{m}, \boldsymbol{n})$ solution to $\text{TAP}(\boldsymbol{G}; \boldsymbol{m}, \boldsymbol{n})$ **(hard)**

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Physics prediction: whp $(\boldsymbol{m}, \boldsymbol{n})$ exists & unique

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Physics prediction ⇒ $(\boldsymbol{m}, \boldsymbol{n})$ " = " $f(\boldsymbol{G})$

$\mathrm{Law}(\boldsymbol{G})$

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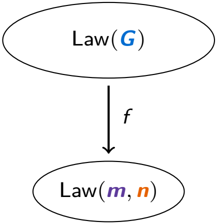

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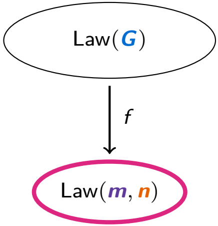

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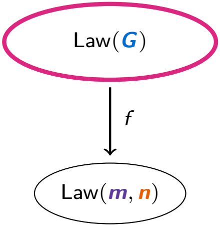

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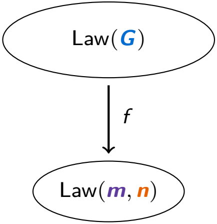

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Also: **1st/2nd mmt works in planted model**

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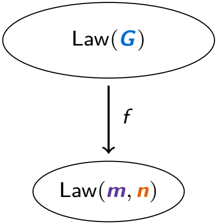


Key idea: TAP planted model

Null model:

- $G \in \mathbb{R}^{M \times N} \sim$ IID $\mathcal{N}(0,1)$ entries
- (m, n) solution to TAP$(G; m, n)$ **(hard)**

Physics prediction: whp (m, n) exists & unique

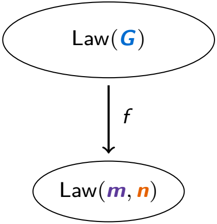
TAP planted model:

- Sample (m, n) from its law (explicit prediction)
- Sample G conditioned on solving TAP$(G; m, n)$ **(easy)**

Under physics prediction: **planted $\approx$ null**!
Also: **1st/2nd mmt works in planted model**

❗ existence/uniqueness of (m, n) is **not proven**.
We will need to justify that planted $\approx$ null.

Physics prediction $\Rightarrow$
(m, n) " = " $f(G)$



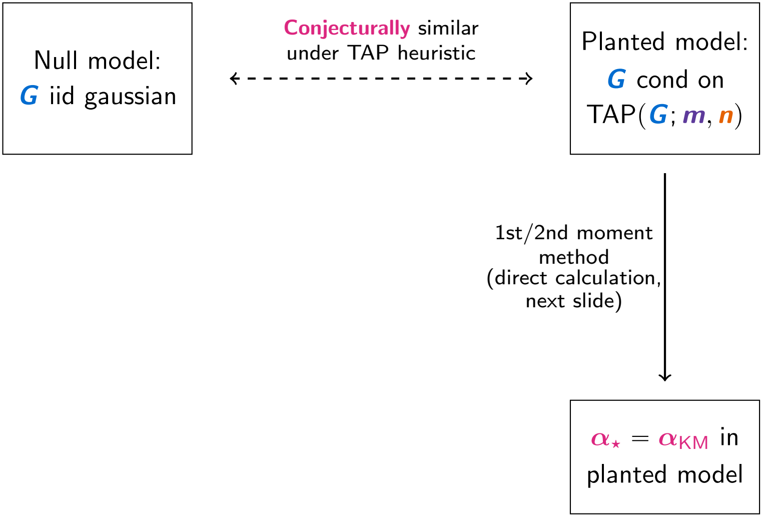
Proof roadmap

Null model:
G iid gaussian

Conjecturally similar under TAP heuristic ⟷

Planted model:
G cond on TAP$(G; m, n)$

Proof roadmap



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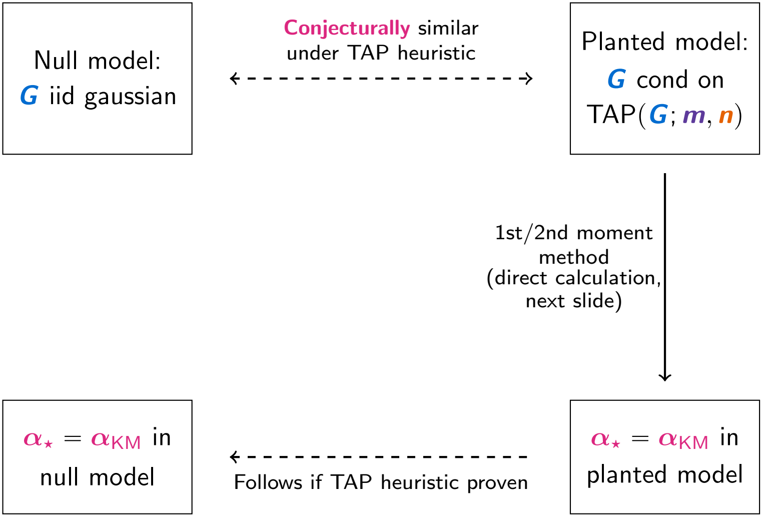

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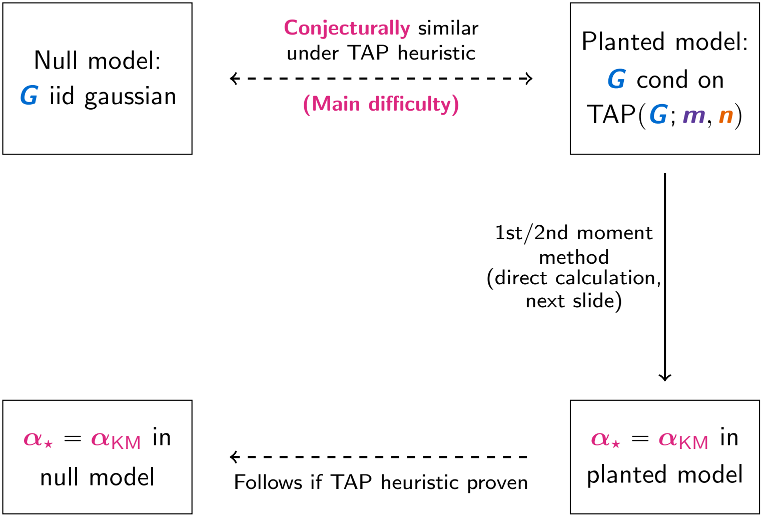

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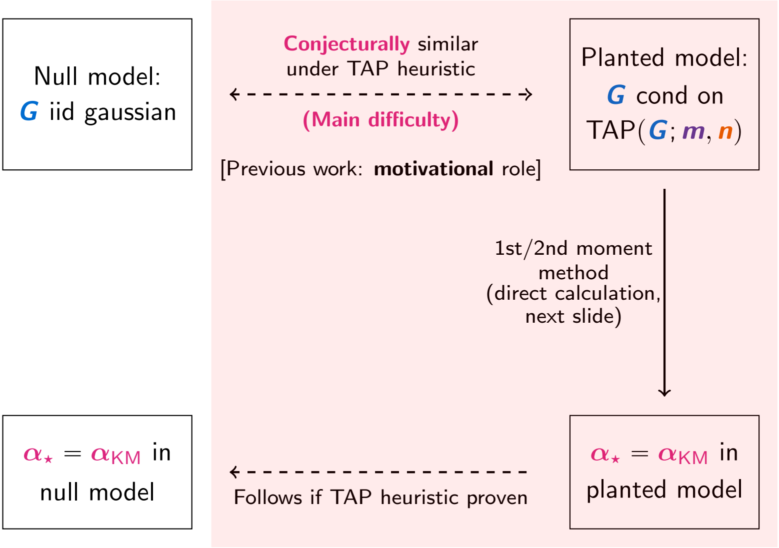

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Conjecturally similar
under TAP heuristic

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(Main difficulty)

[Previous work: **motivational** role]

Planted model:
$\boldsymbol{G}$ cond on
$\mathrm{TAP}(\boldsymbol{G}; \boldsymbol{m}, \boldsymbol{n})$

DS18: lower bd by
1st/2nd mmt on
truncation of $|\boldsymbol{S}|$

1st/2nd moment
method
(direct calculation,
next slide)

$\alpha_\star = \alpha_{\mathrm{KM}}$ in
null model

←- - - - - - - - - - - - - - - - - -

Follows if TAP heuristic proven

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Our work proves this. . .

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1st/2nd moment method (direct calculation, next slide)

$\alpha_\star = \alpha_{\mathrm{KM}}$ in null model

...justifying this implication

Follows if TAP heuristic proven

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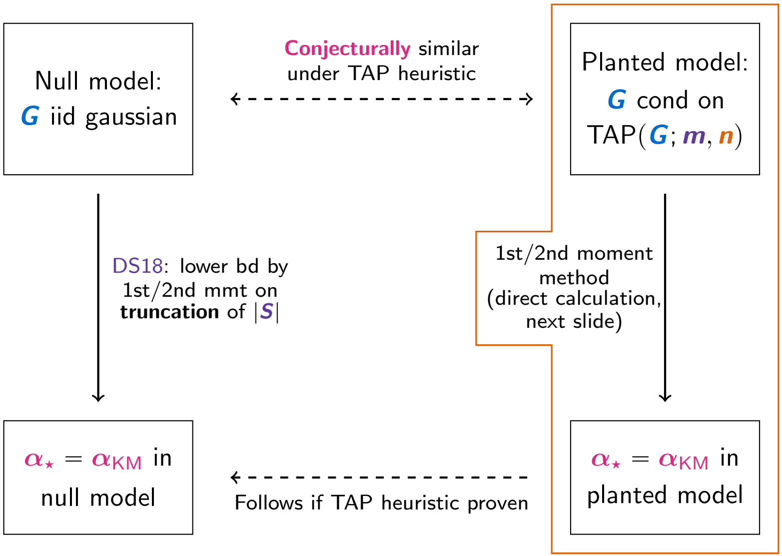
Proof roadmap



1st/2nd moment works in planted model

Recall planted model:

- Sample $(\boldsymbol{m}, \boldsymbol{n})$ from its law
- Sample $\boldsymbol{G}$ conditional on $\dot{F}^{-1}(\boldsymbol{m}) = \frac{\boldsymbol{G}^\top \boldsymbol{n}}{\sqrt{N}} - d\boldsymbol{m}$, $\widehat{F}^{-1}(\boldsymbol{n}) = \frac{\boldsymbol{G}\boldsymbol{m}}{\sqrt{N}} - b\boldsymbol{n}$

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$\Rightarrow$ conditional moments of $|\boldsymbol{S}(\alpha N)|$ remain tractable. For typical $(\boldsymbol{m}, \boldsymbol{n})$,

$$\mathbb{E}[|\boldsymbol{S}(\alpha N)||\boldsymbol{m}, \boldsymbol{n}] \quad \approx \quad \mathbb{E}[|\boldsymbol{S}(\alpha N)|^2|\boldsymbol{m}, \boldsymbol{n}]^{1/2}$$

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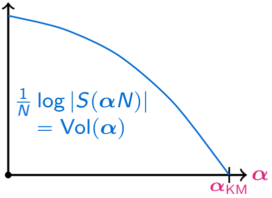

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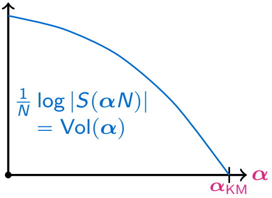

$\Rightarrow$ planted model has capacity α_{KM}

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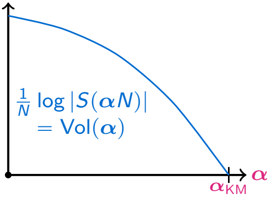
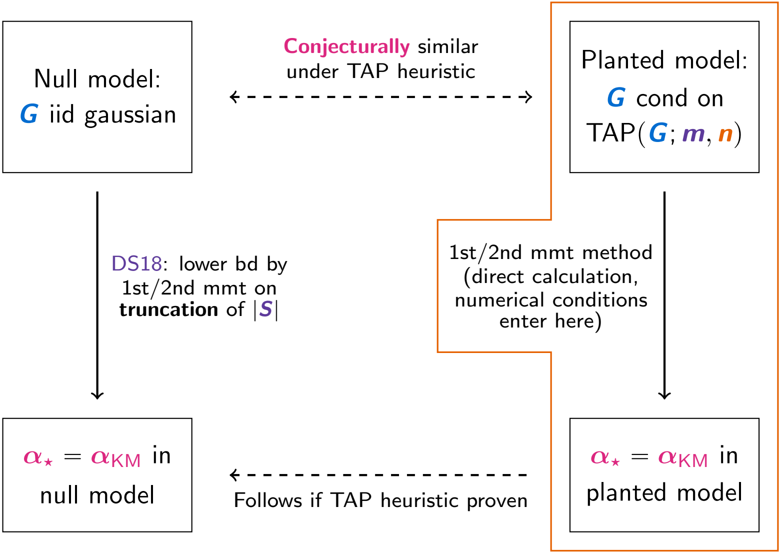


$\Rightarrow$ planted model has capacity α_{KM}

(under our + DS18's numerical conditions)

Proof roadmap



Proof roadmap

Null model: G iid gaussian

Conjecturally similar under TAP heuristic

We now explain how to make this step rigorous.

Planted model: G cond on $\mathrm{TAP}(G; m, n)$

DS18: lower bd by 1st/2nd mmt on **truncation** of $|S|$

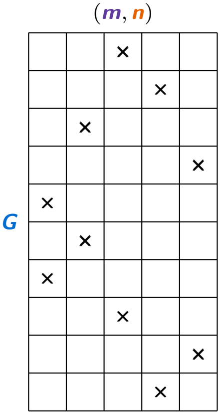
1st/2nd mmt method (direct calculation, numerical conditions enter here)

$\alpha_\star = \alpha_{\mathrm{KM}}$ in null model

Follows if TAP heuristic proven

$\alpha_\star = \alpha_{\mathrm{KM}}$ in planted model

Key issue: linking true and planted models



× indicates $(\boldsymbol{m}, \boldsymbol{n})$ is TAP fixed point of $\boldsymbol{G}$

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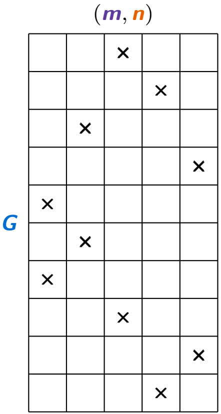

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Null model: random row

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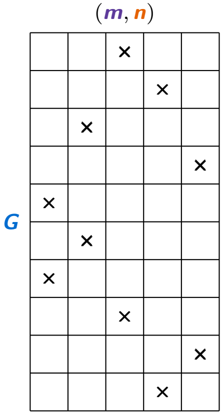

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so null $\approx$ planted

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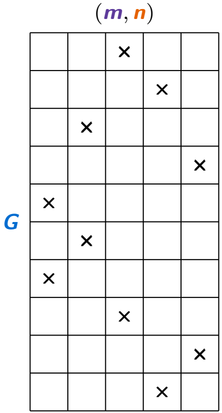

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but. . . **we don't actually know this** ☹

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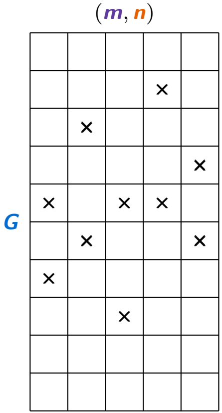

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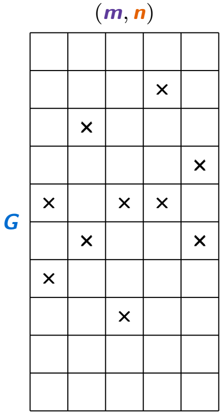
Key issue: linking true and planted models



× indicates (m, n) is TAP fixed point of G

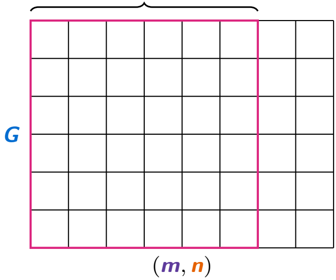
Null model: random row
Planted model: random col, then random × in col

TAP prediction: most rows have exactly one ×
so null ≈ planted

but. . . **we don't actually know this** ☹
⇒ planted / null models can a priori be different

Our work: contiguity of null / planted models

$\boldsymbol{T} = \{\text{“typical”}\ (\boldsymbol{m}, \boldsymbol{n})\}$ (suitably defined set; whp in planted model)



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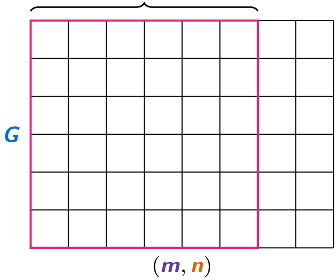

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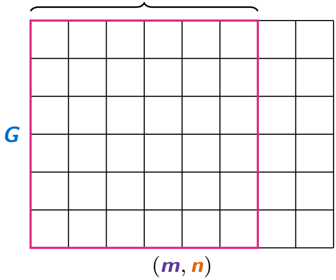

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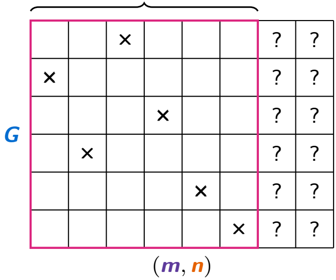

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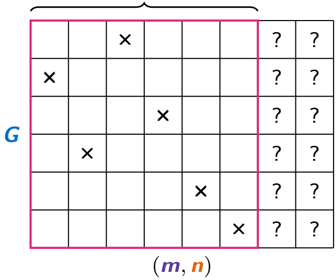

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This shows null $\approx$ planted. Formally,

$$\mathbb{P}_{\text{null}}(\boldsymbol{E}) \leqslant O(1) \cdot \sup_{(\boldsymbol{m},\boldsymbol{n}) \in \boldsymbol{T}} \mathbb{P}_{\text{planted}}(\boldsymbol{E} | \boldsymbol{m}, \boldsymbol{n}) + o(1) \qquad \text{for all event } \boldsymbol{E}$$

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$$\boldsymbol{m}^{k+1} = \dot{F}\left(\frac{\boldsymbol{G}^\top \boldsymbol{n}^k}{\sqrt{N}} - d\boldsymbol{m}^k\right) \qquad \boldsymbol{n}^k = \widehat{F}\left(\frac{\boldsymbol{G}\boldsymbol{m}^k}{\sqrt{N}} - b\boldsymbol{n}^{k-1}\right)$$

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Follows from existing tools to analyze AMP:

- **AMP state evolution** (Bayati Montanari 11, Bolthausen 14, ...)
- **Local concavity of TAP free energy** near late AMP iterates (Celentano Fan Mei 21, Celentano 22, Celentano Fan Lin Mei 23)

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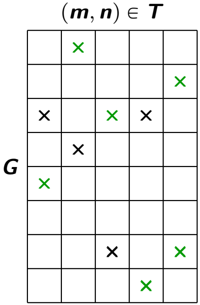

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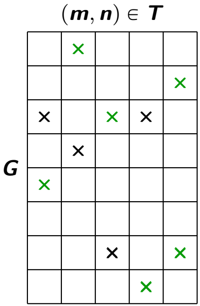

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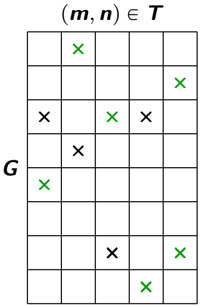

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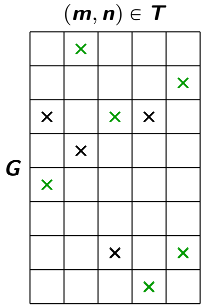

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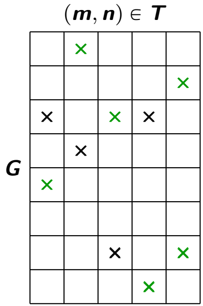

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Uniqueness: AMP returns home in planted model

Remains to show: for $(\boldsymbol{m}, \boldsymbol{n}) \in \boldsymbol{T}$, $\boldsymbol{G}$ conditioned on $\mathrm{TAP}(\boldsymbol{G}, \boldsymbol{m}, \boldsymbol{n})$,

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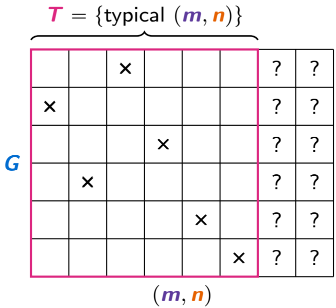
This can be proved by the same AMP state evolution + local concavity of TAP free energy analyses.

Crucially: recall $\mathrm{Law}_{\mathrm{planted}}(\boldsymbol{G} \mid \boldsymbol{m}, \boldsymbol{n})$ remains gaussian. This provides enough structure to adapt these techniques.

Recap: contiguity of null / planted models



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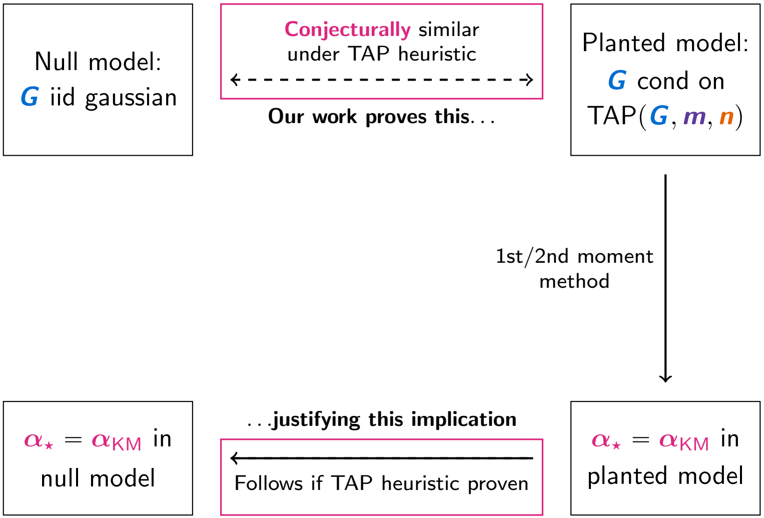

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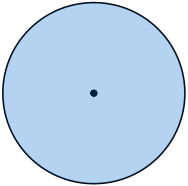

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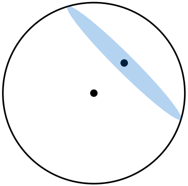

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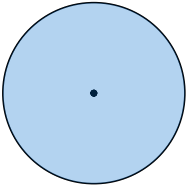

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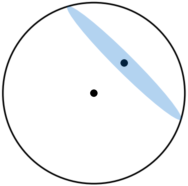

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TAP planting lets you **condition on the random center**, effectively reducing to the mean-zero case. Usages in spin glass sampling:

- High-precision estimation of mean(μ) (H Montanari Pham 24)
- Covariance bound $\|\mathrm{cov}(\mu)\|_{\mathrm{op}} = O(1)$ (H Mohanty Rajaraman Wu 24)

Conclusion

Null model:

- $H \sim \text{Law}(\text{problem})$
- $\boldsymbol{\sigma} \sim \text{Gibbs}(H)$ **(hard)**

Planted model:

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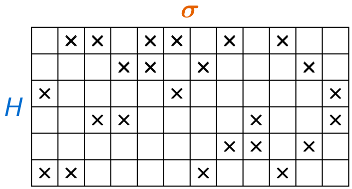

Applications:

- shattering & RS free energy of many models
- spin glass diffusion sampling
- ground state large deviation & 1RSB ground state energy
- capacity of Ising perceptron

Part II: a survey on the overlap gap property

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Outline of part II: a survey on the overlap gap property

Introduction and motivating problems

Overlap gap property: the basics

More OGPs and algorithm classes

Further enhancements

Hardness of finding strict local maxima

Strong low degree hardness

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- **Random perceptron**: for IID $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^M \sim \mathcal{N}(0, I_N)$, $\underset{\text{activation}}{\underset{\uparrow}{\varphi}} : \mathbb{R} \to \mathbb{R}$,

$$H(\boldsymbol{\sigma}) = \sum_{a=1}^{M} \varphi\Big(\frac{(\boldsymbol{\sigma}, \boldsymbol{g}^a)}{\sqrt{N}}\Big)$$

Random optimization problems: motivation

- MLE in statistical tasks, e.g. **tensor PCA**: estimate $\boldsymbol{x}_0 \sim \text{unif}(\mathbb{S}^{N-1})$ from

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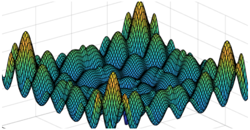

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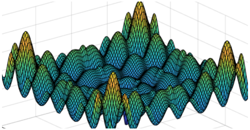

• Random perceptron ↔ loss landscape of neural net on **random data**

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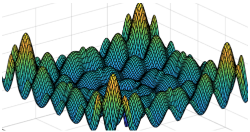

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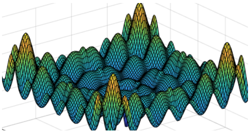

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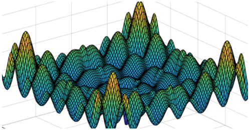

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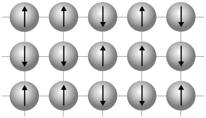
Sampling: algorithmically sample from **Gibbs measure** $\mu_\beta(\boldsymbol{\sigma}) \propto e^{\beta H(\boldsymbol{\sigma})}$. For which β can an efficient algorithm succeed?

Comparison with ferromagnetic Ising model

Ferromagnetic Ising: positive couplings on edges of a graph G



$$H^{\mathsf{Fer}}(\boldsymbol{\sigma}) = \sum_{(i,j)\in E(G)} \sigma_i \sigma_j$$

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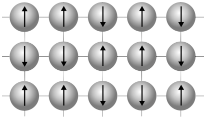

$$H^{\mathrm{Fer}}(\boldsymbol{\sigma}) = \sum_{(i,j)\in E(G)} \sigma_i \sigma_j$$

Main tension between **entropy** and **energy**. For $\mu_\beta(\boldsymbol{\sigma}) = \frac{1}{Z} e^{\beta H^{\mathrm{Fer}}(\boldsymbol{\sigma})}$

- β small $\Rightarrow$ entropy wins, coordinates of $\boldsymbol{\sigma} \sim \mu_\beta$ not aligned
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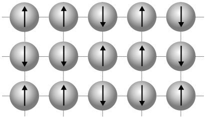

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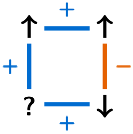
In spin glasses, random $g_{i,j}$ yield **frustration**: can't satisfy all couplings. A priori unclear what ground state looks like.



Comparison with signal recovery

Many similar problems about detecting / recovering a planted signal:

- **Planted clique**: find a k-clique planted in $G(N, 1/2)$
- **Tensor PCA**: recover rank 1 spike planted in gaussian p-tensor
- **Single/multi-index models**: recover $\boldsymbol{W}^*$ from $y_i = f(\boldsymbol{W}^* \boldsymbol{x}_i)$

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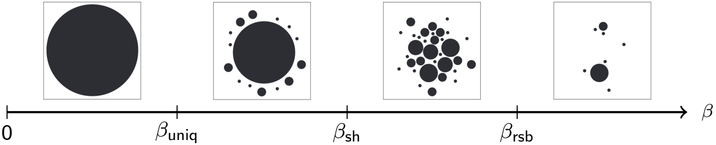
The models we focus on are "pure noise," no planted signal

- **Null models** for signal recovery problems
- Progress can be made "in many directions"
- No notion of sample complexity / SNR

Some possible phases for disordered systems

Predictions of geometric phase transitions + algorithmic implications:



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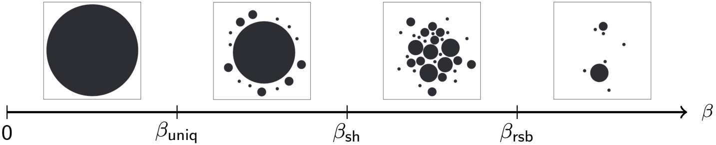

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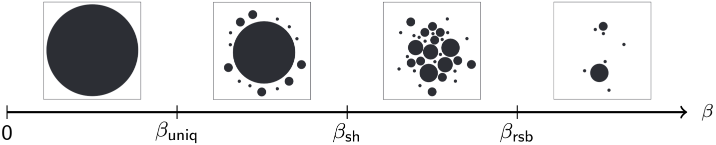

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Does solution geometry have rigorous implications for algorithms?

This talk: overlap gap property

Gamarnik Sudan 14: **solution landscape** properties → rigorous hardness for **stable** algorithms in random optimization / search problems

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(Contrast: for **sampling**, shattering threshold β_{sh} appears to be the fundamental barrier; much recent progress)

Outline of part II: a survey on the overlap gap property

Introduction and motivating problems

Overlap gap property: the basics

More OGPs and algorithm classes

Further enhancements

Hardness of finding strict local maxima

Strong low degree hardness

Where it all started

Max independent set: find a large ind set of Erdős–Rényi $G(N, d/N)$
(d large but fixed, $N \to \infty$)

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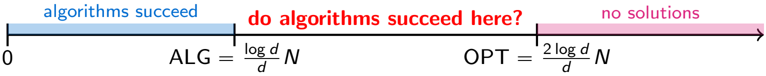

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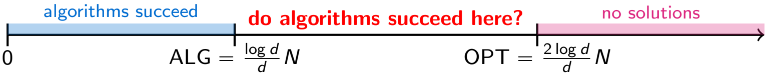

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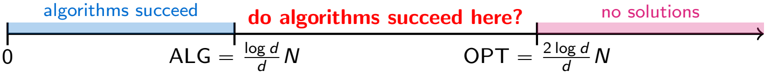

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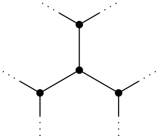

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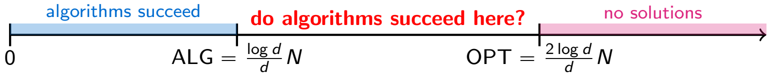

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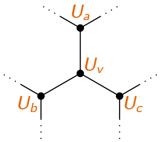
Local algorithm: generate $U_v \sim \text{unif}([0,1])$ at each $v \in G$ (shared randomness)



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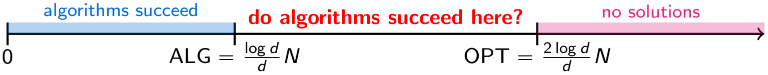

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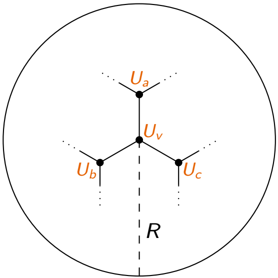


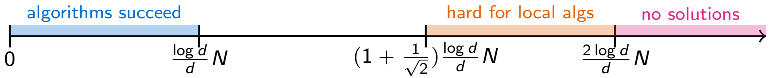
Local algorithm: generate $U_v \sim \text{unif}([0, 1])$ at each $v \in G$ (shared randomness)

At each $v \in G$, decide output $\sigma_v \in \{0, 1\}$ based on only data within R-**neighborhood** of v ($R = O(1)$)

Local algorithms do not reach OPT

Theorem (Gamarnik Sudan 14)

Any $O(1)$-local algorithm does **not** *find an independent set of size $\geqslant (\mathbf{1 + \frac{1}{\sqrt{2}}} + o_d(1))\frac{\log d}{d}N$ in $G(N, d/N)$, whp.*



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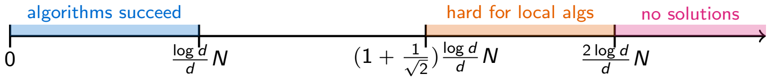

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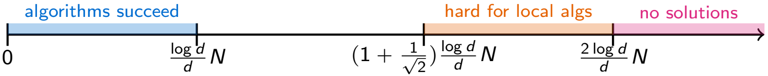

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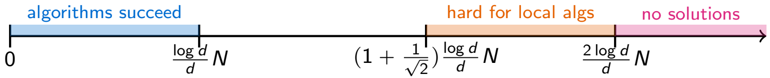

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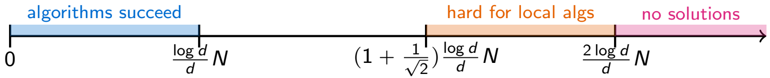

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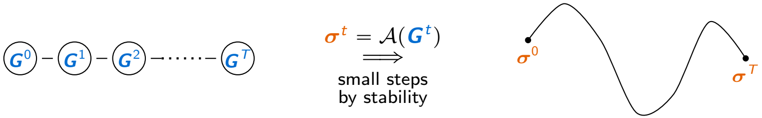

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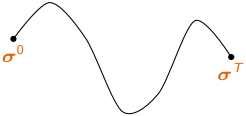

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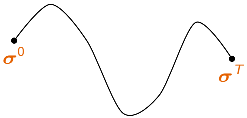

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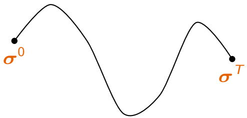

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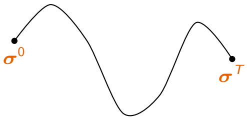

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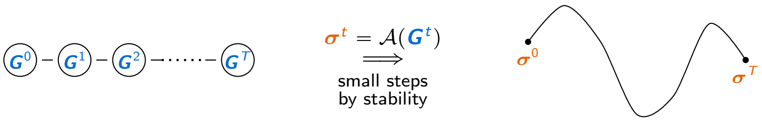

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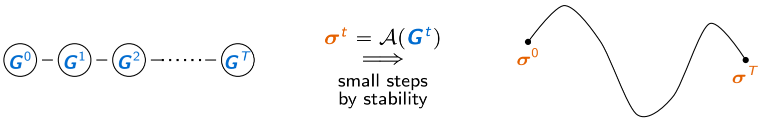

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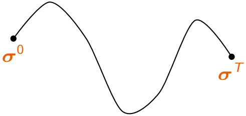
Proof for both: calculate $\mathbb{E}\#\{\text{such } (\sigma, \rho)\} \ll 1$

Local algorithms do not reach OPT

$\sigma^t = \mathcal{A}(G^t)$
$\Longrightarrow$
small steps
by stability



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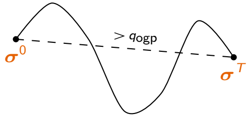

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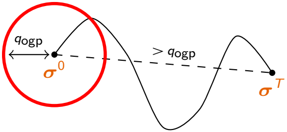

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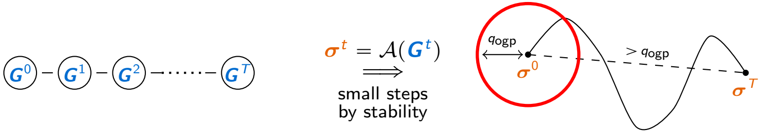

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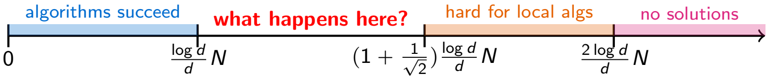
Questions

- Can we show a tighter bound?



- Problems beyond max independent set?

- Algorithm classes beyond local algorithms?

- Finer-grained runtime bounds?

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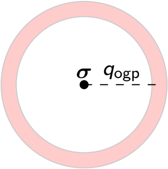

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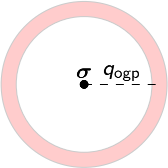

no solutions in red zone

Key distinction: clustering of **most** vs **all** solutions

• Shattering, RSB, etc. concern when **most** solutions cluster/isolated. Algorithms may succeed by finding atypical solutions (Baldassi Ingrosso Lucibello Saglietti Zecchina 15, Abbe Li Sly 21)

• OGP: **all** solutions cluster (even across correlated instances), which implies hardness rigorously

Remarks

OGP uses **geometry** to rule out **stable algorithms**. We hope this is indicative of hardness for all **polynomial time** algorithms.

Known exceptions:

- Random k-XOR-SAT exhibits OGP, but solved by gaussian elimination
- Lattice methods use algebraic structure (Zadik Song Wein Bruna 21)
- Shortest path exhibits OGP but easy (Li Schramm 24)

Outline of part II: a survey on the overlap gap property

Introduction and motivating problems

Overlap gap property: the basics

More OGPs and algorithm classes

Further enhancements

Hardness of finding strict local maxima

Strong low degree hardness

Beyond the classic OGP

Many developments after the classic OGP, following same principle:

- If algorithm succeeds, it can build some constellation of solutions
- But we can show this constellation doesn't exist

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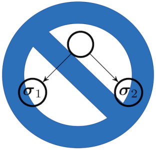

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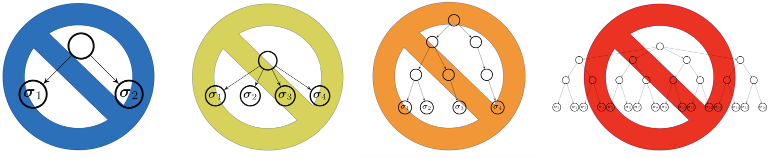
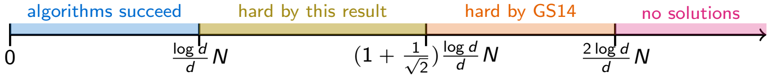


- **Classic** OGP: two points with distance q (Gamarnik Sudan 14)
- **Star** OGP: several points with pairwise distance q (Rahman Virág 17)
- **Ladder** OGP: $\boldsymbol{\sigma}^i$ has distance q to span$(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^{i-1})$ (Wein 21)
- **Branching** OGP: densely branching tree (H Sellke 21)

Star OGP: tight hardness for max independent set

Theorem (Rahman Virág 17)

Any $O(1)$-local algorithm does not find an independent set of size $\geqslant (\mathbf{1} + o_d(1))\frac{\log d}{d}N$ in $G(N, d/N)$, whp.



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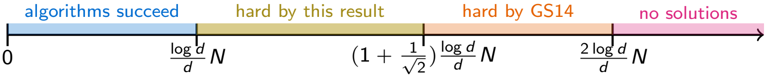
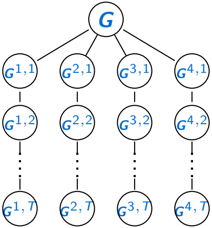


Construct $m = O(1)$ independent resample paths:



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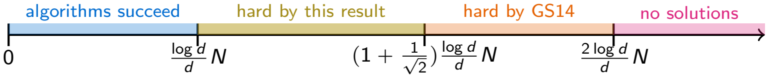

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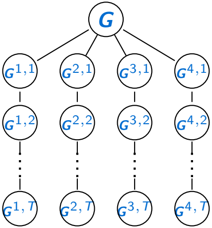

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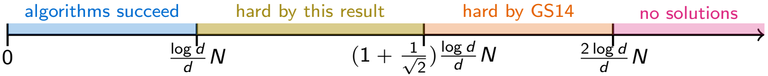

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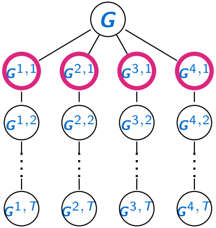

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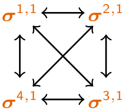

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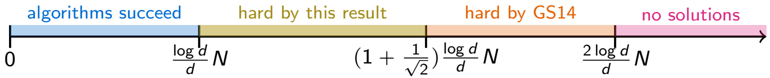

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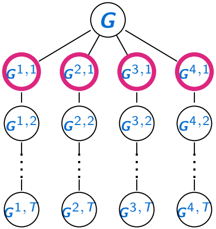

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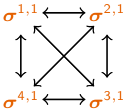

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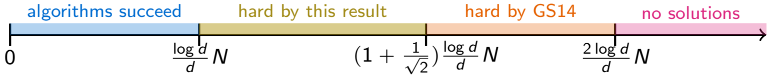

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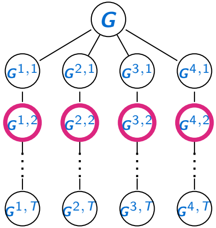

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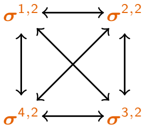

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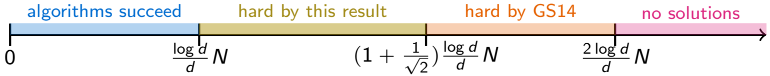

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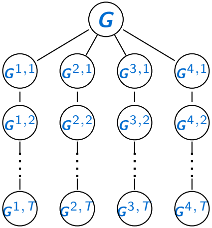

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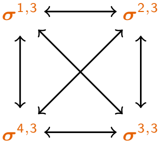

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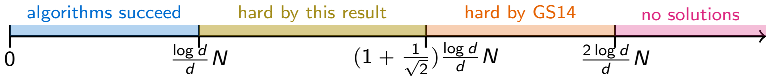

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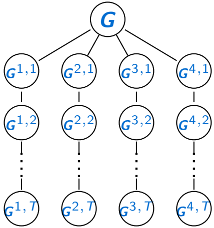

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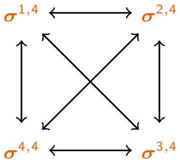

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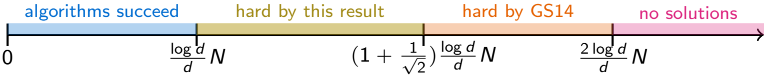

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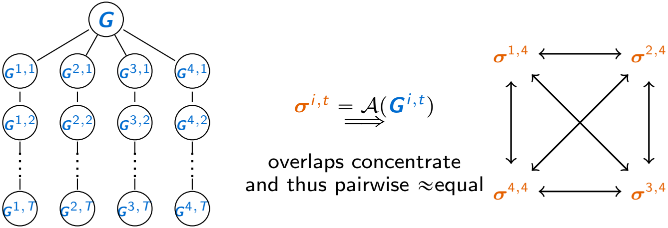

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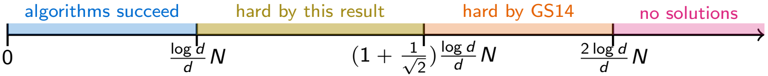

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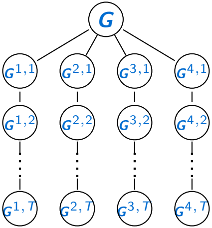

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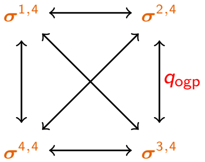

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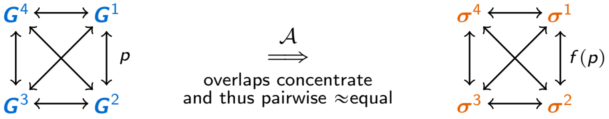
Stable vs concentrated algorithms

This argument uses strongly that overlaps $(\mathcal{A}(G), \mathcal{A}(G'))$ concentrate



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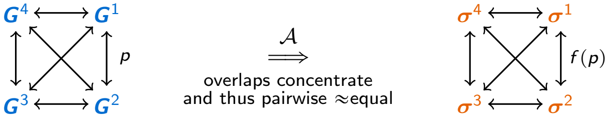

- **Stable** alg: $\mathbb{E}\|\mathcal{A}(G) - \mathcal{A}(G')\|_2^2 \lesssim \varepsilon N$ for $(1-\varepsilon)$-correlated G, G'
- Stable alg is **concentrated** if: $(\mathcal{A}(G), \mathcal{A}(G'))$ concentrates for q-correlated G, G', $\forall q \in [0, 1]$

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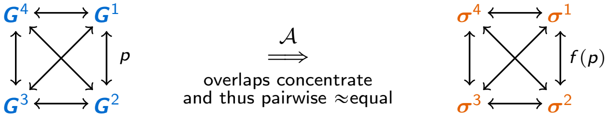

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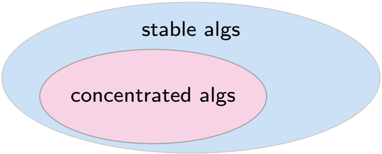

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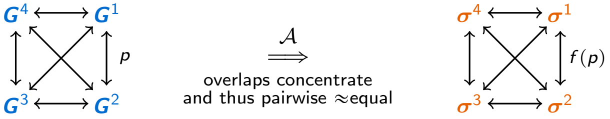

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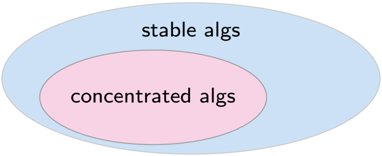

- Concentration $\Rightarrow$ control **all $\binom{m}{2}$ overlaps** among $\mathcal{A}(G^1), \ldots, \mathcal{A}(G^m)$
- Stability $\Rightarrow$ can only use **IVT** considerations to control $\approx m$ **overlaps**.

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Stable algorithms:
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- Gradient descent, Langevin dynamics for $O(1)$ time
- AMP and general $O(1)$ order algorithms for $O(1)$ time
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- Low degree polynomials
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- Gradient descent, Langevin dynamics for $O(1)$ time
- AMP and general $O(1)$ order algorithms for $O(1)$ time
- $O(1)$-local algorithms
- Low degree polynomials
- Low depth circuits

Concentrated algorithms:
$(\mathcal{A}(G), \mathcal{A}(G'))$ concentrates for q-correlated G, G', $\forall q \in [0,1]$

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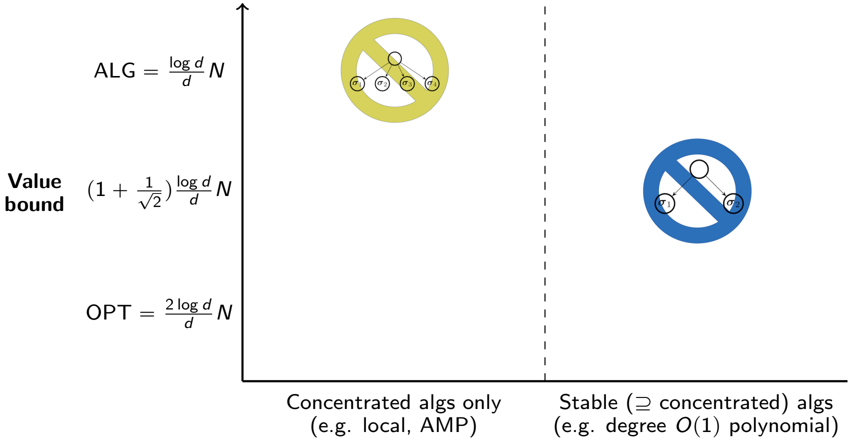
Two classes of OGP hardness proofs: those where **stability is enough**, and those that **only work on concentrated algorithms**

Comparison of OGPs for max independent set



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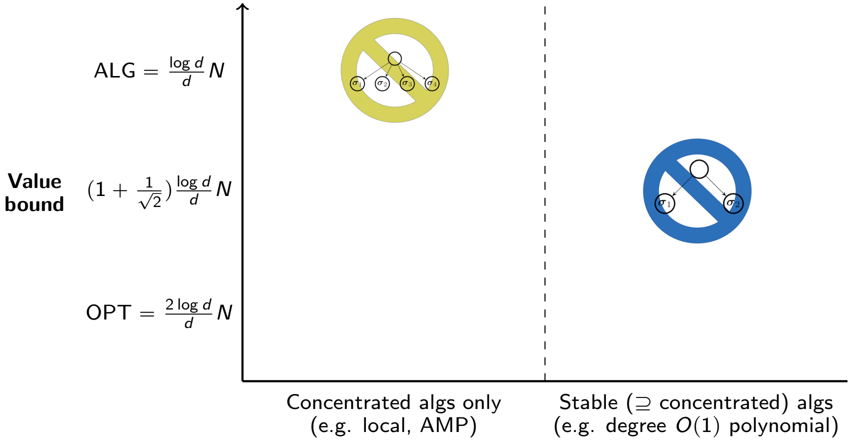

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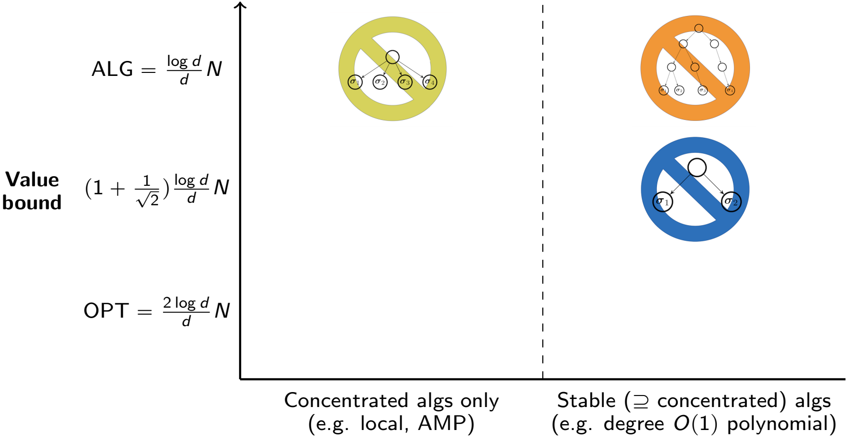

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(But see Buhai Hsieh Jain Kothari 25 for counterexample)

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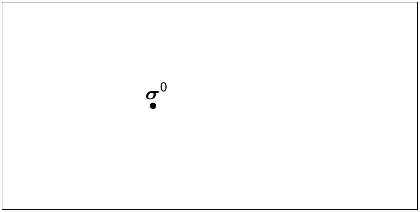

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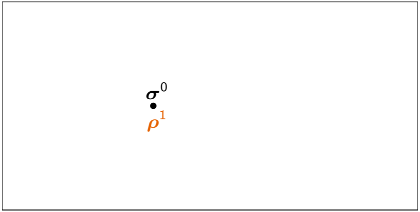

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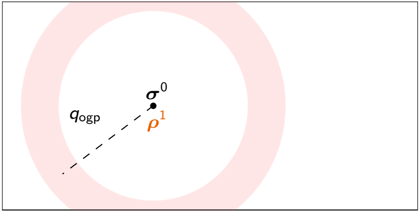

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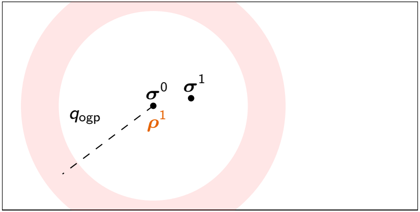

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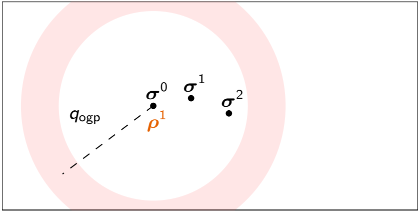

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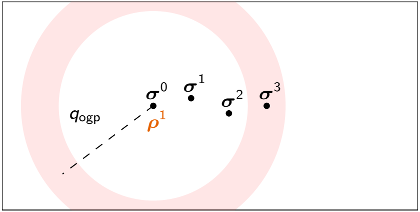

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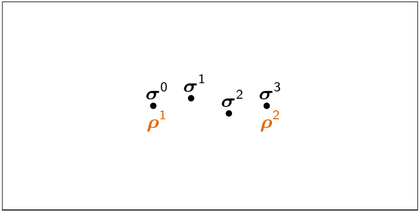

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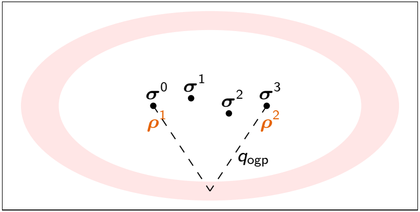

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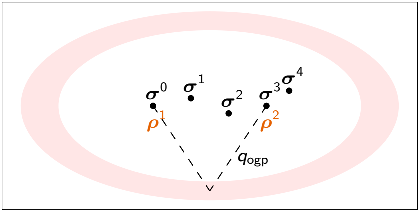

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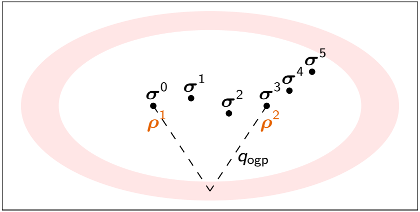

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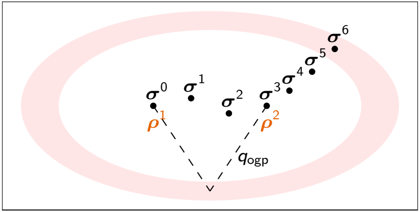

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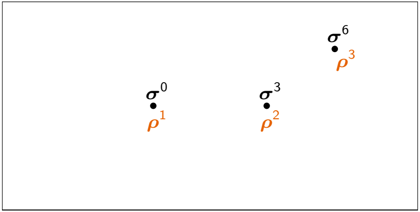

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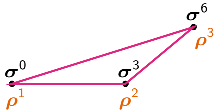
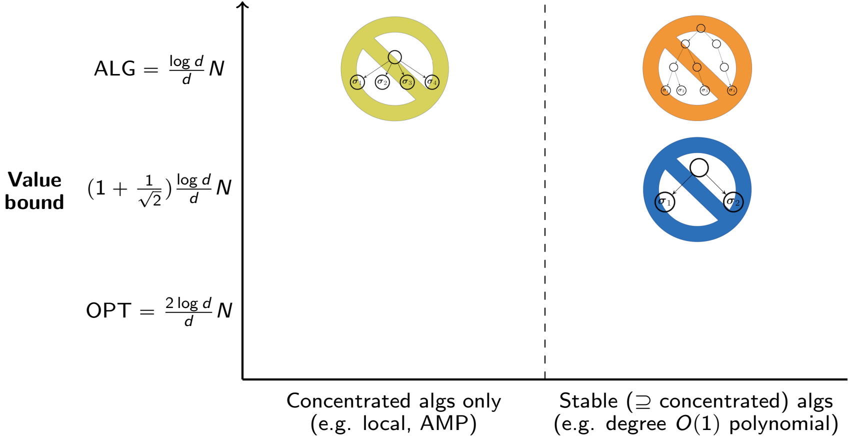


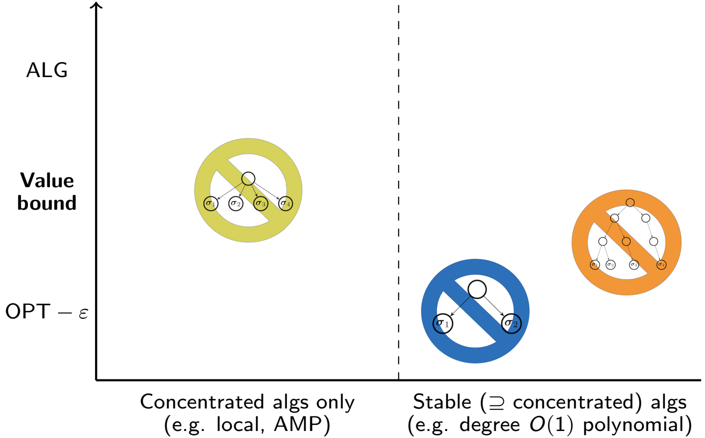
This is the forbidden structure!

Comparison of OGPs for max independent set



What about other problems?

Comparison of OGPs in general



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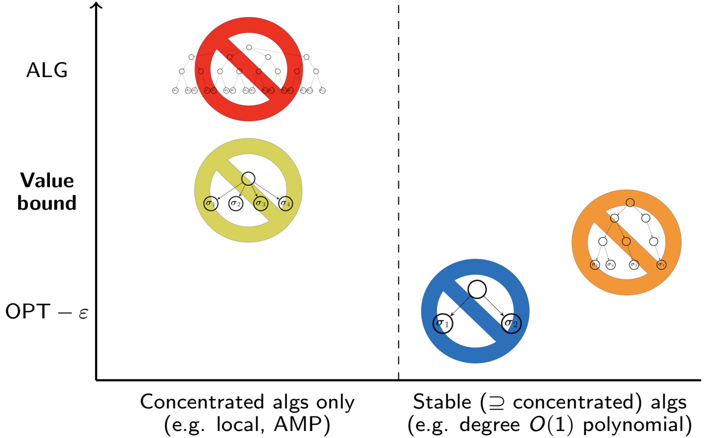

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Another model: mean-field spin glass

Polynomials with IID gaussian coefficients, e.g. random cubic

$$H(\boldsymbol{\sigma}) = \frac{1}{N} \sum_{i,j,k=1}^{N} g_{i,j,k}\, \sigma_i \sigma_j \sigma_k\,, \qquad g_{i,j,k} \overset{IID}{\sim} \mathcal{N}(0,1)$$

More generally, linear combinations of different degrees.

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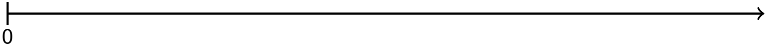

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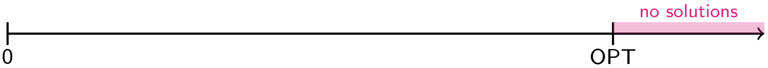

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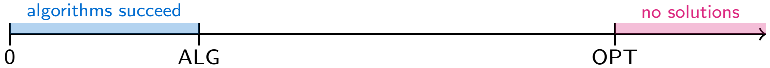

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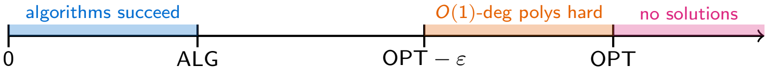

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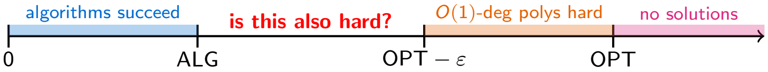

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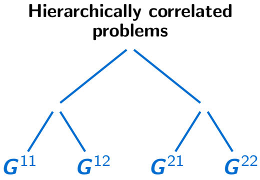

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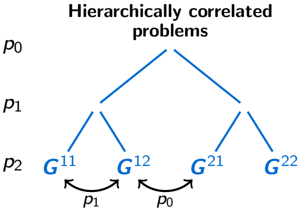

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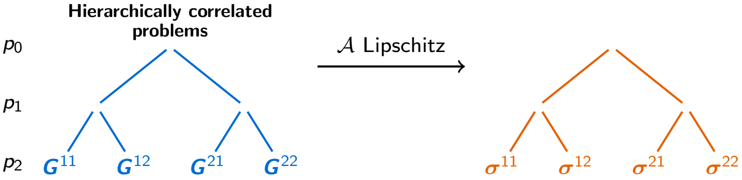

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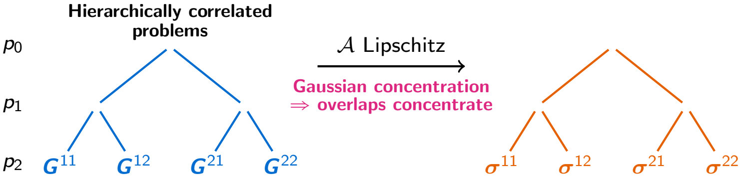

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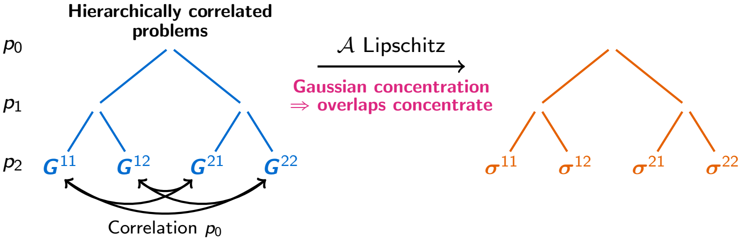

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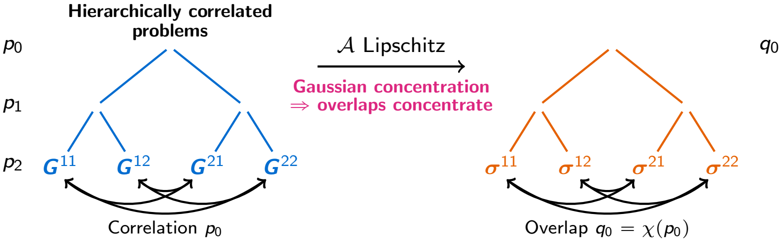

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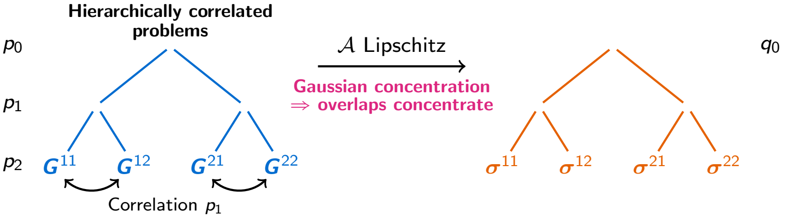

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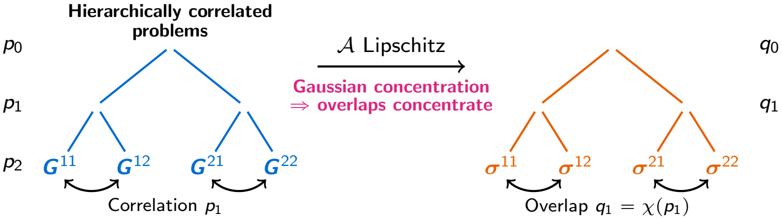

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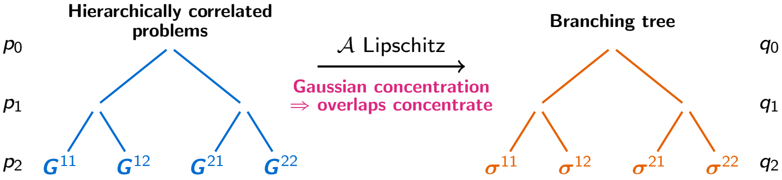

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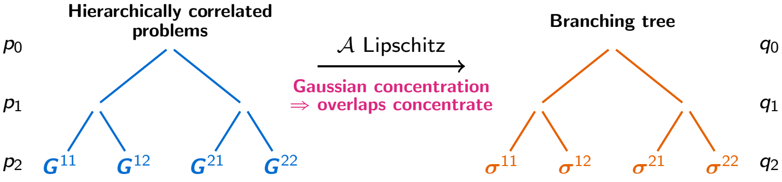

Forbidden structure: branching tree of $\boldsymbol{\sigma}^i$ each with value $\geqslant$ ALG $+ \varepsilon$

Geometric description of algorithmic threshold

(Lipschitz) algorithmic threshold is the supremal $\boldsymbol{E}$ whose super-level set

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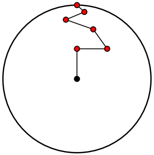

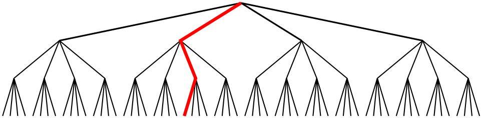

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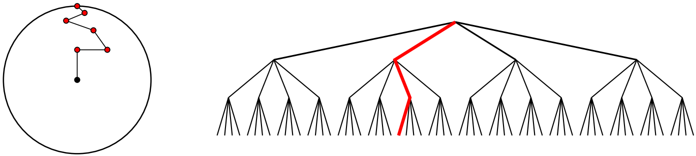

- Hardness: any Lipschitz algorithm can be made to output such a tree.

Exact algorithmic thresholds in many problems

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- Random systems of polynomial equations (Montanari Subag 24)
- Largest average submatrix / subtensor (Gamarnik Li 16, Bhamidi Gamarnik Gong 25)
- Random perceptron (Montanari Zhou 24, H Sellke Sun 25^{+})

Outline of part II: a survey on the overlap gap property

Introduction and motivating problems

Overlap gap property: the basics

More OGPs and algorithm classes

Further enhancements

Hardness of finding strict local maxima

Strong low degree hardness

Further enhancements

OGP methodology:

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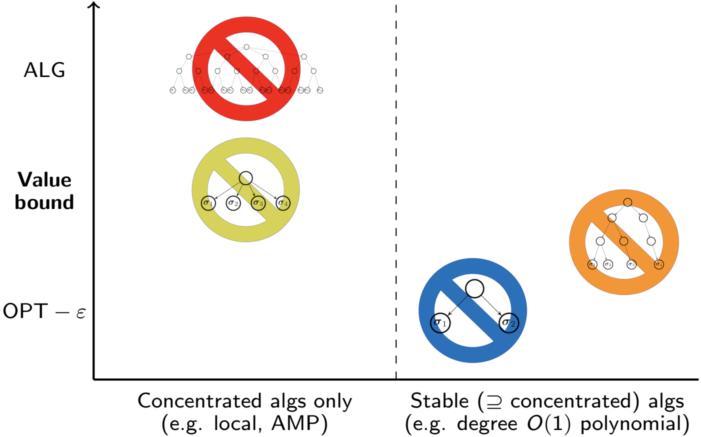
Next few slides: enhancements to step 1. More clever ways to force algorithm to **build a simple constellation**

Ramsey trick



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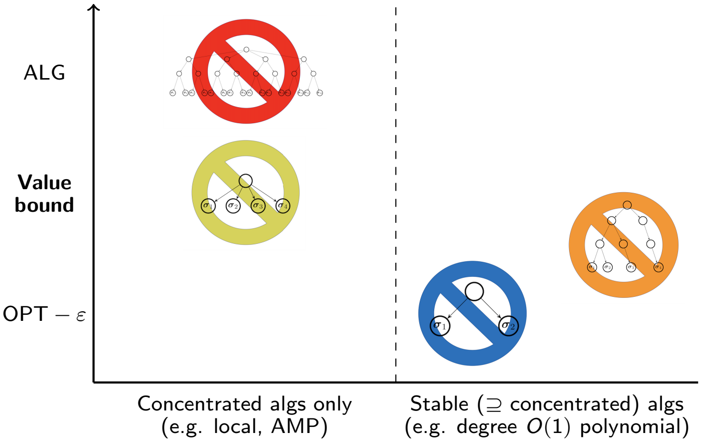

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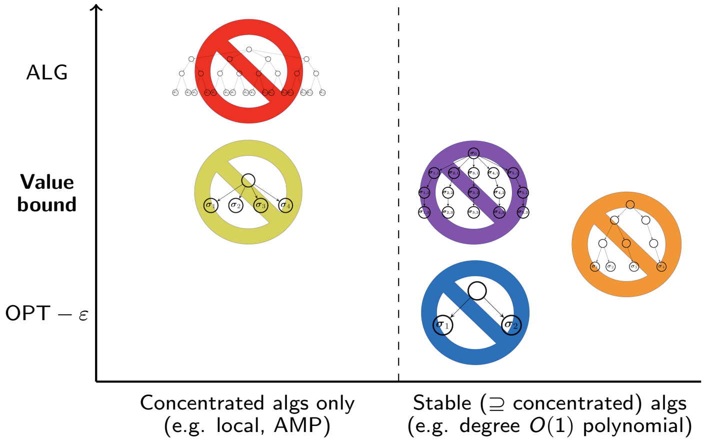

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Symmetric Ising perceptron satisfies star OGP

Problem satisfies **star** OGP at $\alpha \gtrsim \kappa^2 \log \frac{1}{\kappa}$. That is, for some m, q_{ogp}

- Sample any (possibly correlated) $\boldsymbol{G}^1, \ldots, \boldsymbol{G}^C$ where $C = O(1)$

Symmetric Ising perceptron satisfies star OGP

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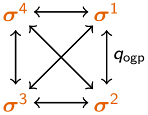

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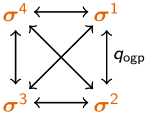
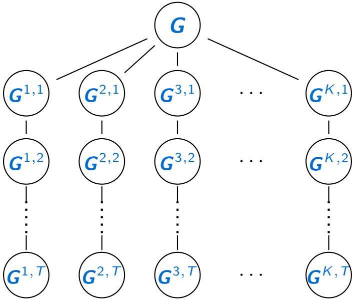


But ... how to construct this structure with a **stable** algorithm?

All we know: for $(1-\varepsilon)$-correlated $\boldsymbol{G}, \boldsymbol{G}'$, $\|\mathcal{A}(\boldsymbol{G}) - \mathcal{A}(\boldsymbol{G}')\|$ small whp

Ramsey-theoretic construction of forbidden structure

Construct K independent resample paths (K, $T = O(1)$, K large)



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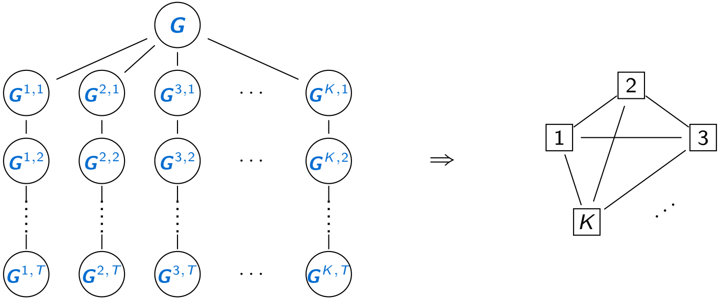

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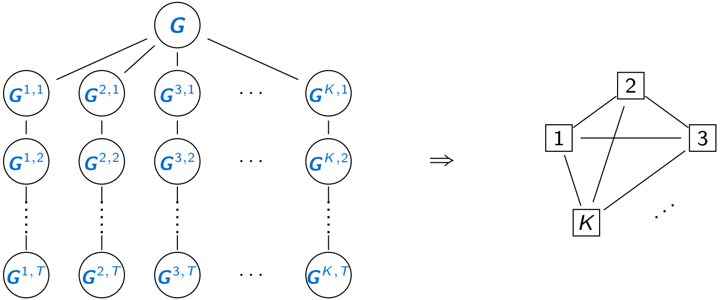

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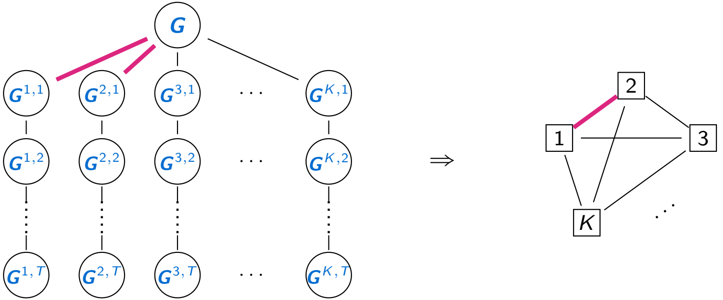

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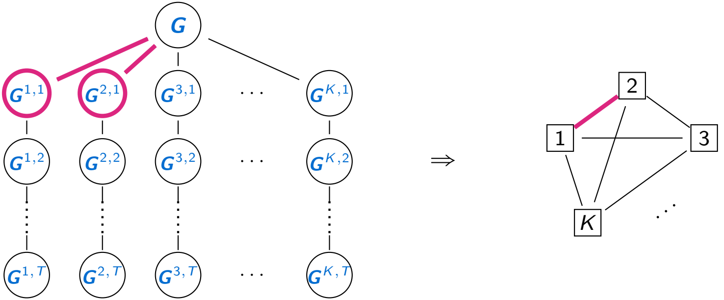

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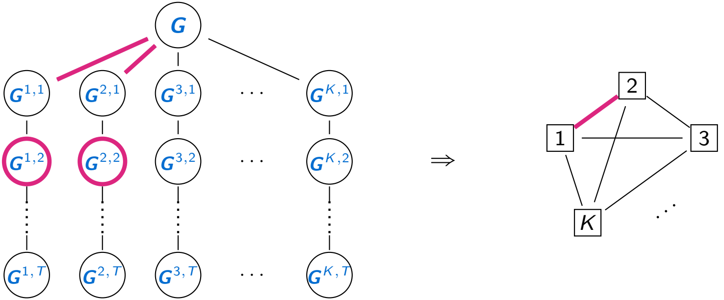

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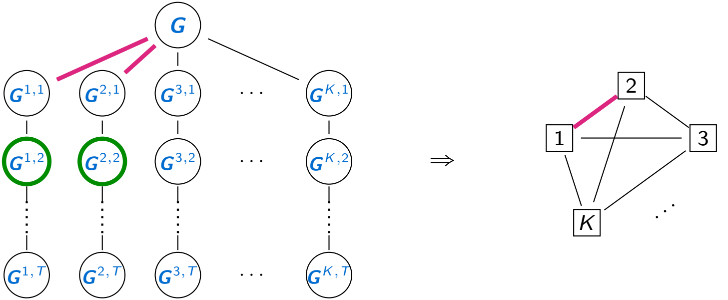

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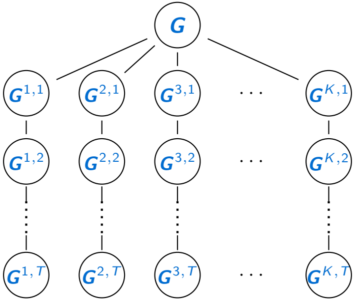

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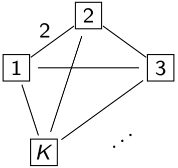

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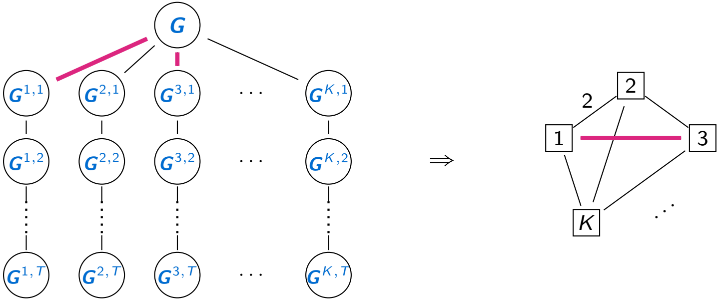

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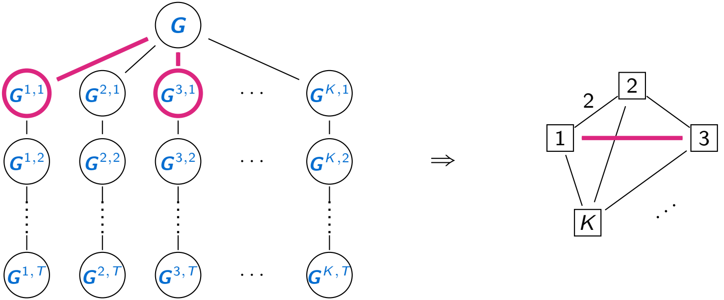

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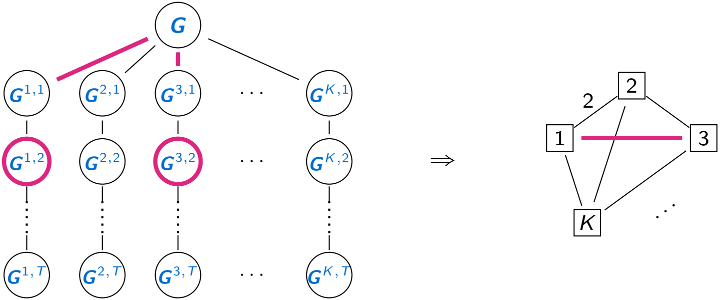

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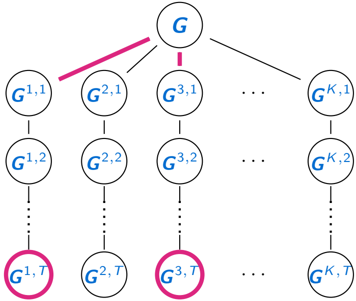

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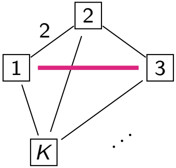

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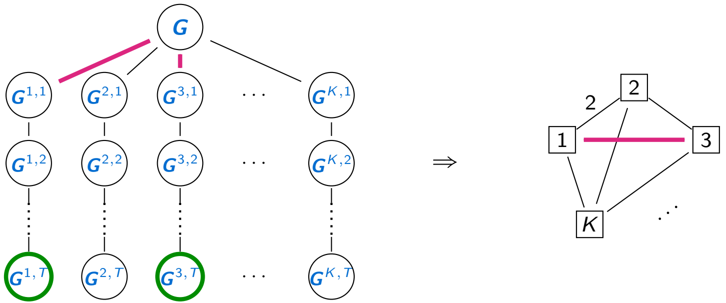

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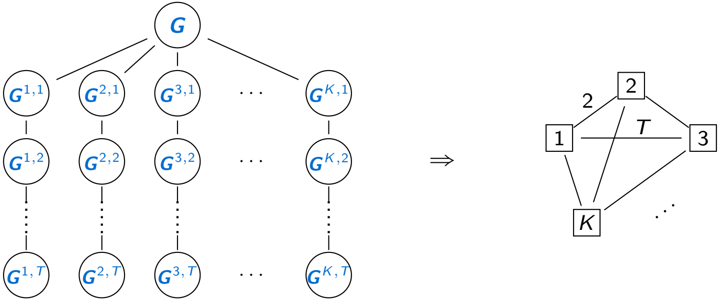

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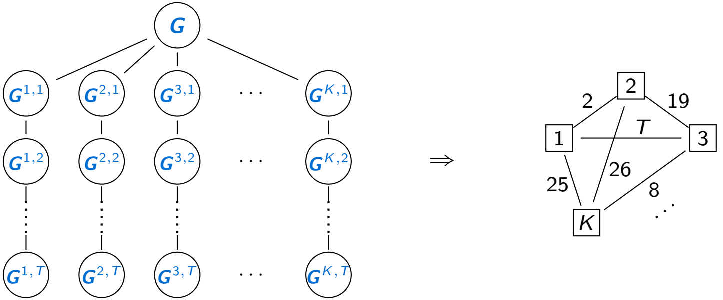

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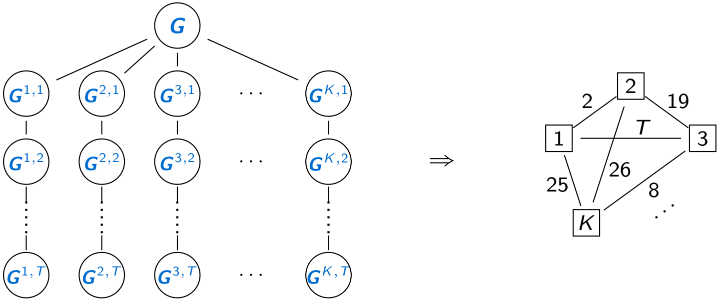

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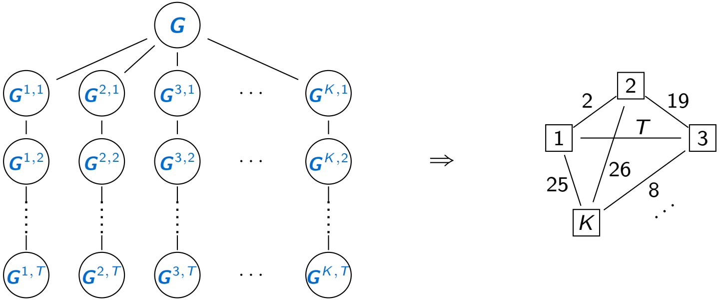

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Q: can we show separation between **online** and **general** algorithms?

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For $M/N = \alpha$, $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^N \overset{\text{IID}}{\sim} \mathcal{N}(0, I_M)$, find $\boldsymbol{\sigma} \in \{\pm 1\}^N$ such that

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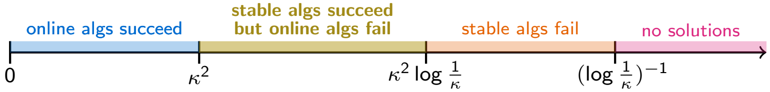
Theorem (Gamarnik Kızıldağ Perkins Xu 23)

Online algorithms cannot beat $\alpha_{\text{online}} \lesssim \kappa^2$



Online OGP

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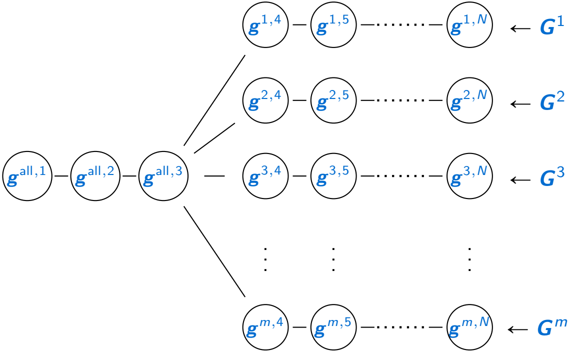

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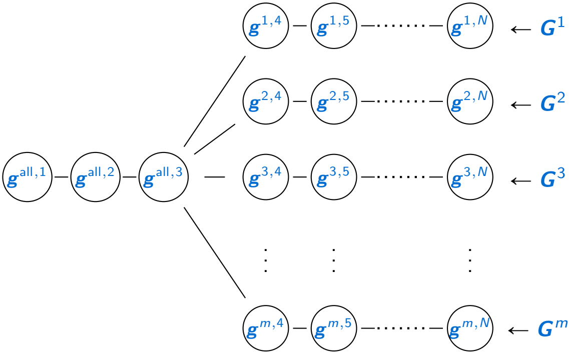

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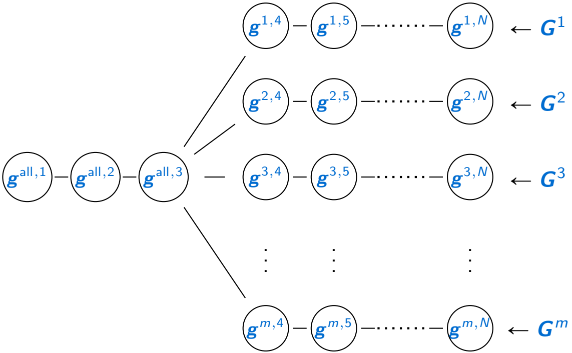

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$\Rightarrow$ easier to show this doesn't exist in solution landscape

Outline of part II: a survey on the overlap gap property

Introduction and motivating problems

Overlap gap property: the basics

More OGPs and algorithm classes

Further enhancements

Hardness of finding strict local maxima

Strong low degree hardness

Can algorithms find strict local maxima?

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Behrens Arpino Kivva Zdeborová 22, Minzer Sah Sawhney 24 conjecture:
All efficient algorithms fail to find a stable local max

Notion of strict local max: gapped states

SK model: Hamiltonian $H : \{\pm 1\}^N \to \mathbb{R}$ defined by

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Q: can an efficient algorithm find a gapped state?

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There exists an explicit $\gamma_c > 0$ such that

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Low degree heuristic $\Rightarrow$ suggests failure of any $e^{o(N)}$ time algorithm!

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If $\boldsymbol{\sigma}^0$ *is a* γ*-gapped state of* H^0 **depending only on** H^0, *then*

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Contrast with earlier OGPs: there **can** exist γ-gapped states $(\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1)$ of (H^0, H^1) with $\Delta(\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1) \in [\varepsilon^3 N, c(\gamma)N]$.

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If $\boldsymbol{\sigma}^0$ is a γ-gapped state of H^0 **depending only on** H^0*, then*

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Contrast with earlier OGPs: there **can** exist γ-gapped states $(\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1)$ of (H^0, H^1) with $\Delta(\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1) \in [\varepsilon^3 N, c(\gamma)N]$.

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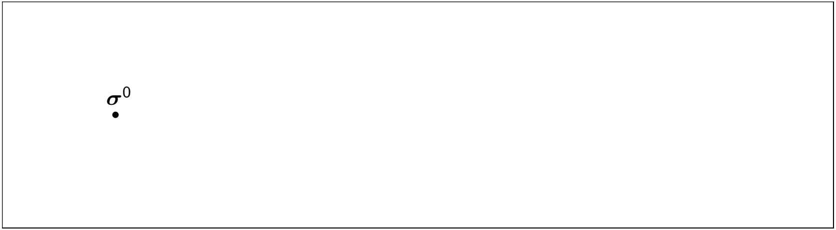

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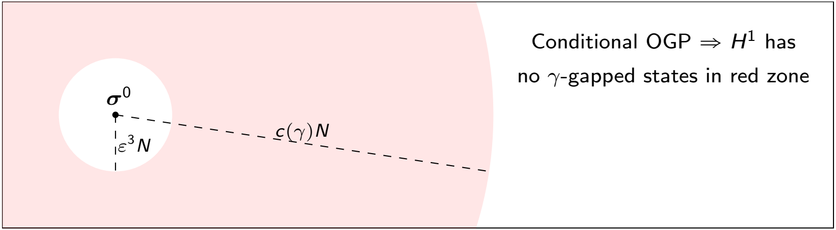

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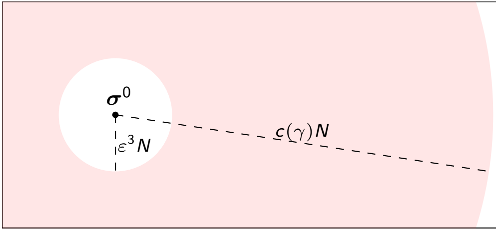

Conditional OGP $\Rightarrow H^1$ has no γ-gapped states in red zone

Stability of $\mathcal{A} \Rightarrow \boldsymbol{\sigma}^1$ has Hamming distance $\leqslant c(\gamma)N$ to $\boldsymbol{\sigma}^0$

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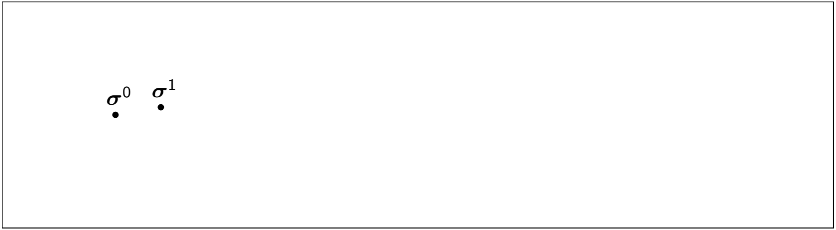

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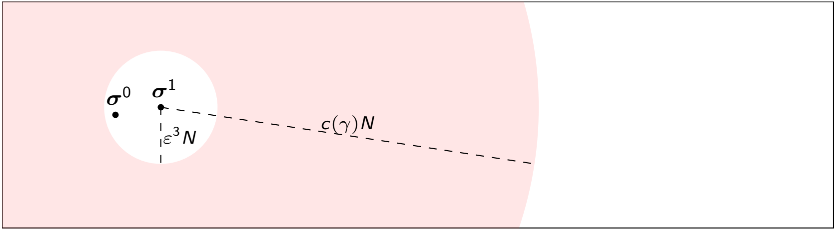

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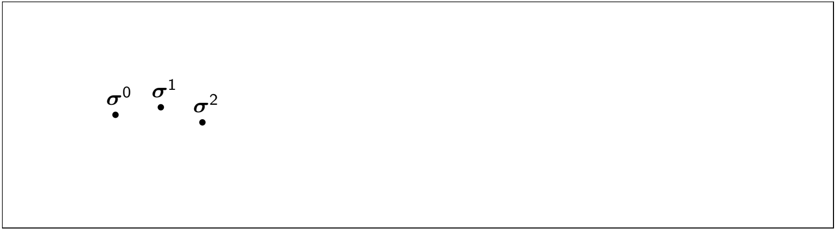

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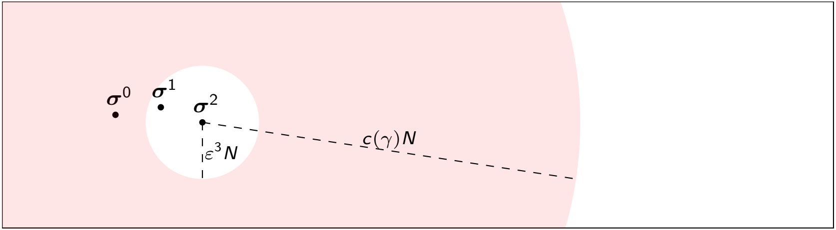

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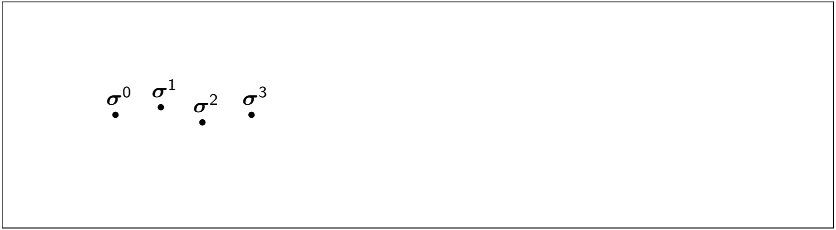

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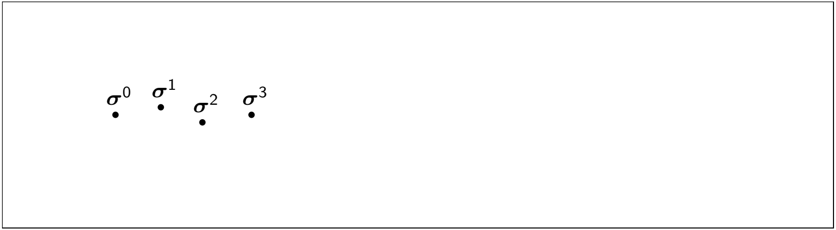

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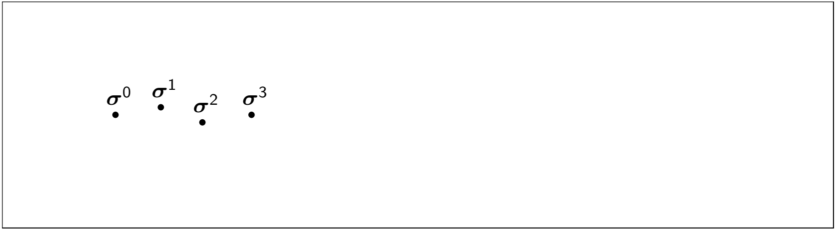

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Not possible because $H^0, H^{1/\varepsilon^2}$ nearly independent!

Hardness for Langevin dynamics on spherical models

Consider mixed p-spin glass

$$H(\boldsymbol{\sigma}) = \sum_{p \geqslant 2} \frac{\gamma_p}{N^{(p-1)/2}} \langle \boldsymbol{G}^{(p)}, \boldsymbol{\sigma}^{\otimes p} \rangle , \qquad \boldsymbol{G}^{(p)}_{i_1,\ldots,i_p} \overset{IID}{\sim} \mathcal{N}(0,1)$$

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Theorem (H Sellke 25)

For any $\gamma > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}(\textit{Low-temperature Langevin finds } (\gamma, \delta)\textit{-stable well in } O(1) \textit{ time}) \leqslant e^{-cN}$$

Outline of part II: a survey on the overlap gap property

Introduction and motivating problems

Overlap gap property: the basics

More OGPs and algorithm classes

Further enhancements

Hardness of finding strict local maxima

Strong low degree hardness

A closer look at the OGP methodology

Following does not occur simultaneously:

- Algorithm $\mathcal{A}$ solves all $H^1, \ldots, H^T$ in the correlated ensemble
- Outputs $\boldsymbol{\sigma}^i = \mathcal{A}(H^i)$ form the desired constellation
- This constellation does not exist in solution space of $(H^1, \ldots, H^T)$

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Union bound: $p_{\text{solve}} \leqslant 1 - 1/T$ ☹

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This issue quietly plagued the OGP literature for years. Numerous works prove $p_{\text{solve}} \leqslant 1 - e^{-\boldsymbol{D}}$ for degree $\boldsymbol{D}$ polynomials:

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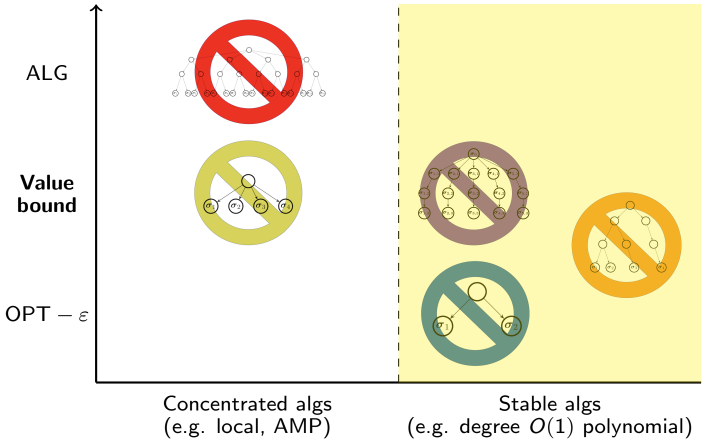
Open problem in Dec 2024 AIM workshop: *Low degree polynomial methods in average-case complexity*

Strong low degree hardness

We give general method to overcome this issue, for **all stability-based** OGPs



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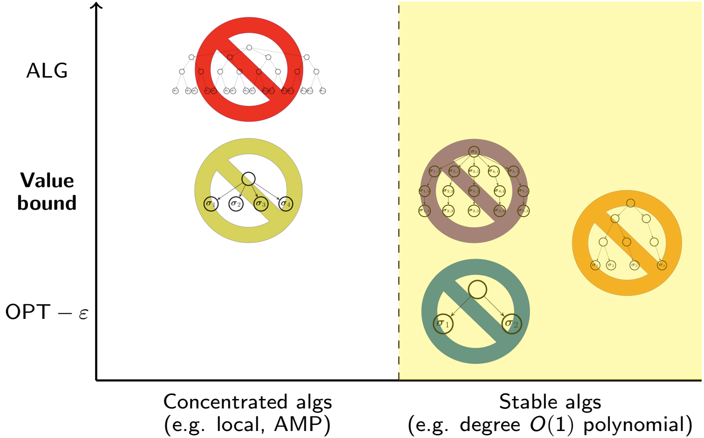

Actually, show $p_{\text{solve}} = o(1)$ for degrees **much larger** than $O(1)$.

Strong low degree hardness

Theorem (H Sellke 25, informal)

If a stability-based OGP obstruction holds with probability $1 - p_{\mathrm{ogp}}$, *then*

$$\mathbb{P}(\textit{a degree } \boldsymbol{D} = \widetilde{o}(\log \tfrac{1}{p_{\mathrm{ogp}}}) \textit{ algorithm succeeds}) = o(1)$$

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This is tight: $\boldsymbol{D} = O(N)$ / time $e^{O(N)}$ can brute force!

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(and in max-clique: $\boldsymbol{D} = O(\log^2 N)$ / time $e^{O(\log^2 N)}$ can brute force)

Strong low degree hardness: proof ideas

Let's revisit **ladder** OGP: consider Markovian sequence of Hamiltonians

$$H^0 \to H^1 \to \cdots \to H^T$$

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(H^i, H^{i+1}) is $(1-\varepsilon)$-correlated. Following doesn't occur simultaneously:

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- $\|\mathcal{A}(H^i) - \mathcal{A}(H^{i+1})\|$ small for all i
- no forbidden structure in soln space of $(H^0, \ldots, H^T)$ (prob $1 - p_{\text{ogp}}$)
  (e.g. $1 - e^{-cN}$)

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Strong low degree hardness: initial attempt

We can iterate this dyadically!

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Doesn't yet imply $p_{\mathrm{solve}} = o(1)$ ☹

Strong low degree hardness via dyadic Jensen

do dyadic Jensen on merged event Solve&Stab$(0, \ldots, T)$:

$$\Big\{\mathcal{A} \text{ solves } H^0, \ldots H^T \text{ and } \|\mathcal{A}(H^i) - \mathcal{A}(H^{i+1})\| \text{ small for } 0 \leqslant i \leqslant T-1\Big\}$$

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(more generally, $\boldsymbol{D} = o(\log \frac{1}{p_{\text{ogp}}})$ if $\mathbb{P}(\nexists$ forbidden structure$) = 1 - p_{\text{ogp}}$)

Strong low degree hardness from star OGP

Theorem (H Sellke 25)

If a star OGP holds with probability $1 - p_{\mathrm{ogp}}$, *then*

$$\mathbb{P}\Big(\textit{a degree } \boldsymbol{D} = o(\log \tfrac{1}{p_{\mathrm{ogp}}} / \log\log \tfrac{1}{p_{\mathrm{ogp}}}) \textit{ algorithm succeeds}\Big) = o(1)$$

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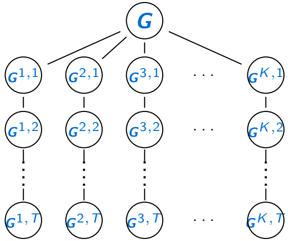
Construct $K \gg 1$ independent resample paths



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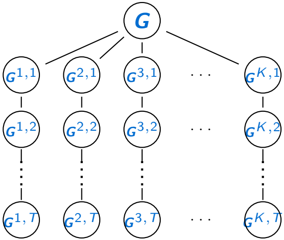

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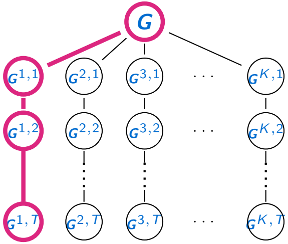

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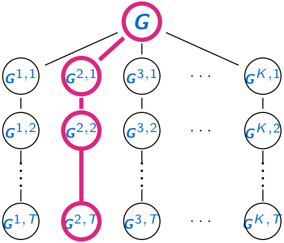

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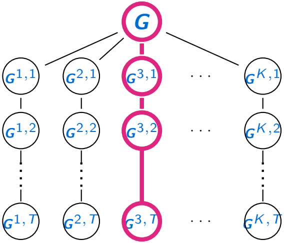

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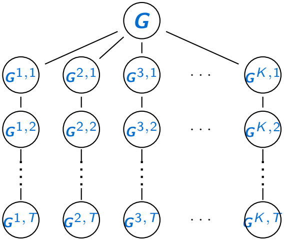

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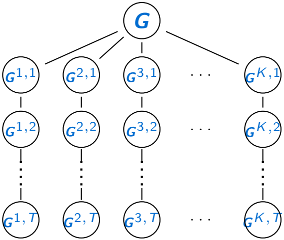

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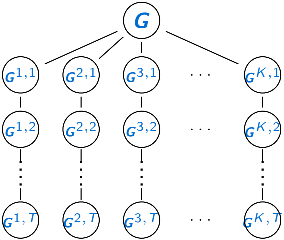

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But on this event, Ramsey trick constructs OGP structure! Contradiction.

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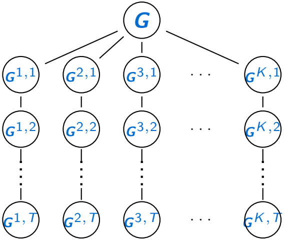
$$\mathbb{P}\Big(\text{a degree } \boldsymbol{D} = o(\log \tfrac{1}{p_{\text{ogp}}} / \log\log \tfrac{1}{p_{\text{ogp}}}) \text{ algorithm succeeds}\Big) = o(1)$$

Construct $K \gg 1$ independent resample paths



$\text{Solve\&Stab}^{(k)} = \{$on k-th arm, $\mathcal{A}$ solves all & all steps stable$\}$

As before: $\mathbb{P}(\text{Solve\&Stab}^{(k)}) \geqslant \mathbb{P}(\text{Solve\&Stab(one step)})^T$

Suppose $p_{\text{solve}} = \Omega(1)$. For K large, can show:

$$\mathbb{P}\left(\text{Solve\&Stab}^{(k)} \text{ holds for } \geqslant \boldsymbol{(Tm)^T} \text{ arms}\right) = \Omega(1)$$

But on this event, Ramsey trick constructs OGP structure! Contradiction.

SLDH for number partitioning problem

NPP: given $g_1, \ldots, g_N \overset{IID}{\sim} \mathcal{N}(0,1)$, find $\boldsymbol{\sigma} \in \{\pm 1\}^N$ minimizing

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This is sharp **for all** $1 \ll \boldsymbol{D} \ll N$: deg $\boldsymbol{D}$ achieves $2^{-\tilde{\Omega}(\boldsymbol{D})}$ by brute force.

Relation to shortest path OGP

Li Schramm 24: shortest path on $G(N, \frac{C \log N}{N})$ satisfies OGP but is easy
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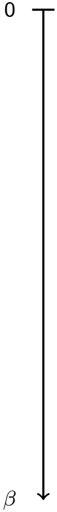
Possible reconciliation: $p_{\text{ogp}} = N^{-\omega(1)}$ necessary for "genuine" hardness

Phase diagram for sampling

For sampling from spin glass Gibbs measure $\mu_\beta(\boldsymbol{\sigma}) \propto e^{\beta H(\boldsymbol{\sigma})}$:



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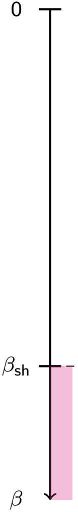

Gibbs measure shatters; sampling predicted hard.
Rigorous hardness for **stable** samplers.
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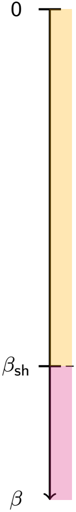

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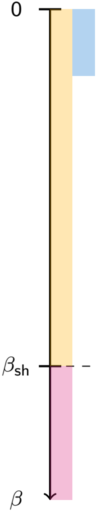

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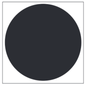

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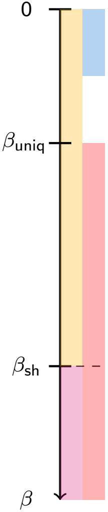

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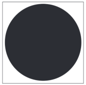

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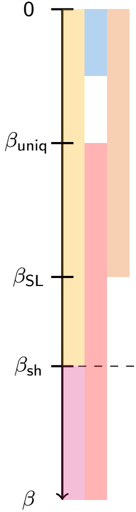

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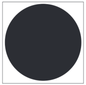

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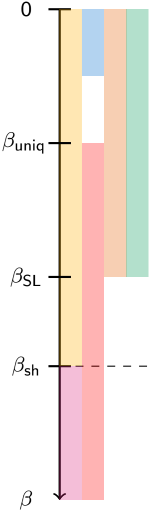

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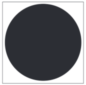

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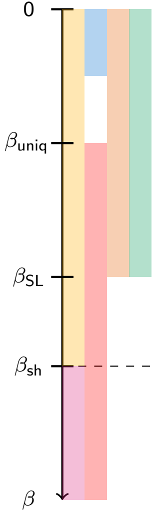

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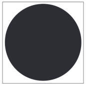

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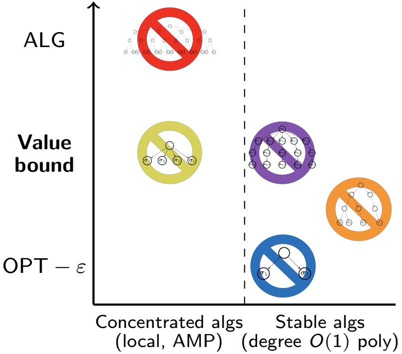
Open problem: sample for $\beta \in (\beta_{\mathsf{SL}}, \beta_{\mathsf{sh}})$

Conclusion

OGP is a powerful geometric framework for computational limits in random search / optimization problems.



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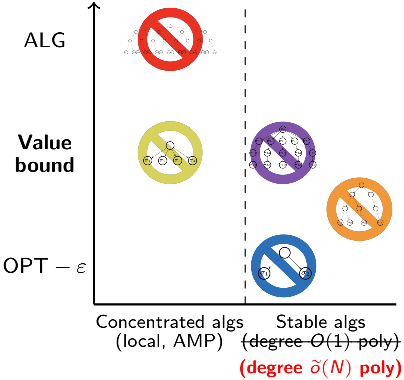

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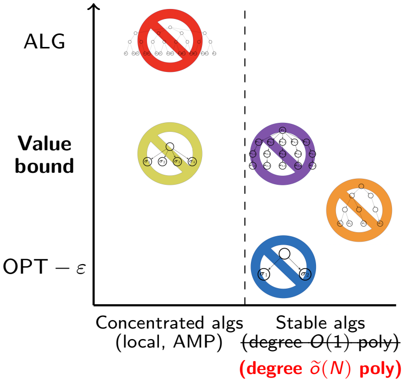

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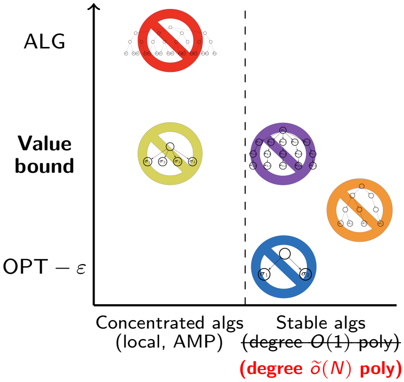

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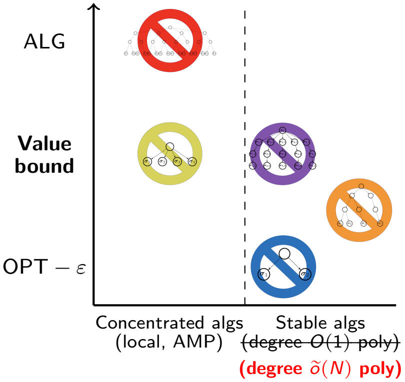

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