The MMSE of the Planted Subgraph Problem

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Parametric Estimation and MMSE

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  exact computation of limiting MMSE for some high-d models!
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This talk

MMSE for Planted Subgraph model: a **combinatorial and sublinear** prior.

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Question

Are *all* subliner-sparsity examples exhibiting the AoN phenomenon?

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Question

A general theory for the MMSE curves of planting an arbitrary subgraph?

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- $\mathrm{MMSE}_n(p)$ is a polynomial-in-p of degree n...

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- **Proofs** via bounding MMSE: upper bound (key: minimax duality) lower bound via *Bayesian proof* of the fractional Kahn-Kalai conjecture [MNWS**Z**'22].

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- Subgraph Kahn-Kalai conjecture *still open*...

MMSE curve: guessing the thresholds a la Kahn and Kalai

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- *Idea*: whenever **no** $S \subseteq H, |S| \leq q|H|$ satisfies $H \setminus S \subseteq G_0 \sim G(n,p)$.

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Theorem [LPR**Z**'25]

For any weakly dense H, there exists $q_0 = 0 < q_1 < \ldots < q_M = 1$ s.t.

- for $i = 0$, if $p \geq (1 + o(1))\phi_{q_0}$, $\mathrm{MMSE}_n(p) = 1 - o(1)$.
- for $i = 0, 1, \ldots, M - 1$,
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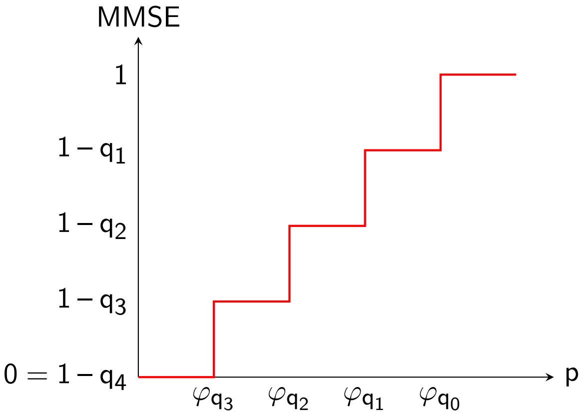
- For all H, the **subgraph Kahn-Kalai threshold** $p_{KK}(H)$ is the **weak recovery threshold**! (a.k.a. **condensation threshold**!)

Pictorial representation

Fix a weakly dense $H = H_n$. The for large enough n:



Refining the picture: MMSE characterization v2

Onion Decomposition of H

Input $H, i = 1, S_0 = \emptyset$.

1. Let $S_i = \arg\max_{S_{i-1} \subseteq S \subseteq H} |S|/v(S)$ (densest subgraph containing S_{i-1})
2. Unless $H \setminus S_i = \emptyset$ repeat step 1 for $i \leftarrow i + 1$.

Output: $S_0 = \emptyset \subseteq S_1 \subseteq S_2 \subseteq \ldots \subseteq S_M = H$.

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(Refined) Theorem [LPR**Z**'25]

For any weakly dense H, let $q_i = |S_i|/|H|, i = 1, \ldots, M$ for S_i o.d. of H

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- $\phi_{q_i} = n^{-v(S_i \setminus S_{i-1})/|S_i \setminus S_{i-1}|}, i = 1, \ldots, M.$
- The $q_i, \phi_{q_i}, i = 1, \ldots, M$ can be computed in poly-time in $|H|$.
  (Leveraging an elegant LP relaxation [Cha'00]!)

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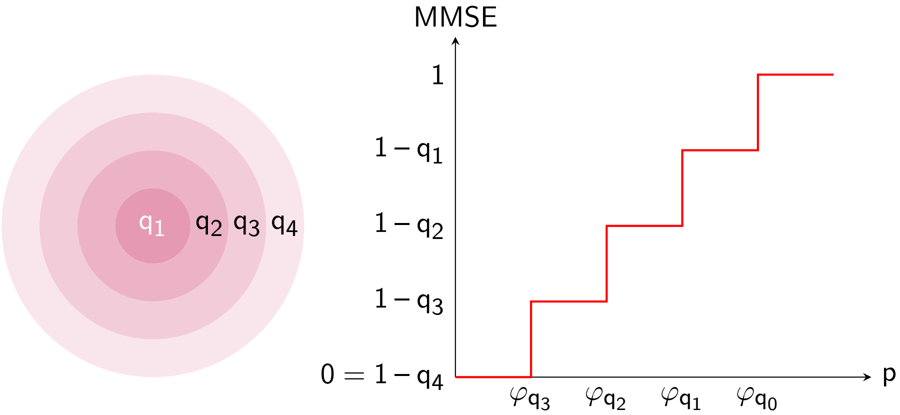

MMSE for Planted Clique (revisited)

Let n vertices, $\mathcal{PC}$ a random k-clique and $G = \mathcal{PC} \cup G_0, G_0 \sim \mathcal{G}(n, p)$.
Then, if say $k = 2\log_2 n$,

$$\lim_n \mathrm{MMSE}_n := \frac{2}{k(k-1)} \mathbb{E}[\|\mathbf{1}(\mathcal{PC}) - \mathbb{E}[(\mathbf{1}(\mathcal{PC})|G]\|_2^2] = \begin{cases} 1 & p > 1/2 \\ 0 & p < 1/2 \end{cases}$$

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  MMSE jumps from $1 - o(1)$ to $o(1)$ at $p = 1/2$.

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Let n vertices, $\mathcal{H} = C_1 \cup C_2$ disjoint union of random k_1-clique and a k_2-clique and $G = \mathcal{H} \cup G_0, G_0 \sim \mathcal{G}(n, p)$.
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This work

For any planted subgraph problem w/ $\mathsf{H} = \mathsf{H}_n$ weakly dense ($|\mathsf{H}| \gg v(\mathsf{H}) \log v(\mathsf{H})$) we **characterize** for large n the MMSE curve.

- The MMSE curve for large n is a piecewise constant function with discontinuities given, up to $1 + o(1)$ error, by variants of the so-called *subgraph Kahn-Kalai threshold* of the graph H. **New stats meaning** to the subgraph Kahn-Kalai thresholds!
- We characterize for each p *which subgraphs* of H are recoverable (*onion decomposition*).
- Both the *onion decomposition* and the MMSE curve can be computed in *polynomal-time* in $|\mathsf{H}|$.
- *Corollary*: AoN happens iff the graph H is balanced [MNWSS**Z**'22].
- **Proofs** via bounding MMSE: upper bound (key: minimax duality) lower bound via *Bayesian proof* of the fractional Kahn-Kalai conjecture [MNWS**Z**'22].

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Thank you!!