2 parts: [redacted]

(1) Refresher of the equations.

(2) Proof ideas.

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(1) Recall we fix $i = xN$ & want to describe $\{\sigma_i^t\}_{t=0}^T \cup \{h_i^t\}_{t=1}^T$.

We have:

$$\{\sigma_i^t\}_{t=0}^T \cup \{h_i^t\}_{t=1}^T \xrightarrow{d} \mathrm{Law}\left(\{\sigma_x^t\}_{t=0}^T \cup \{h_x^t\}_{t=1}^T\right).$$

- Moreover, prop. of chaos so that empirical measure converges.
- Law described by sampling ~~proc~~ procedure: $\sigma_x^0 \sim \mathrm{Unif}\{\pm 1\}$. Sample seeds $\{U^t\}_{t=1}^T$ iid $\mathrm{Unif}[-1,1]$. Let for $t = 1, \dots, T$,

$$\begin{cases} \sigma_x^t = c(h_x^t, U^t) \\ h_x^t = G_x^t + \langle (\sigma_x^0, \dots, \sigma_x^{t-1}), f^t(x) \rangle \end{cases}$$

[Glauber: $c(h,u) = \mathrm{sign}(\tanh(\beta h) - u)$]

Here: $\{G_x^t\}_{t=1}^T$ is a G.P. w. cov. $\Sigma^{(T)}(x) \in \mathbb{R}^{T \times T}$; $f^t \in \mathbb{R}^t$, $t = 1, \dots, T$.

Our equations: closed diff. eq. ~~for~~ for f^t, $t = 1, \dots, T$ and $\Sigma^{(T)}$.

$$\Sigma^{(T)}(x) = \int_0^x C_{1:T,1:T}(y)\,dy + \int_x^1 C_{0:T-1,0:T-1}(y)\,dy.$$

Here, for $0 \le s, t \le T$, $C_{st}(y) = \mathbb{E}\,\sigma_y^s \sigma_y^t$. [keep this eqn on the board.]

$$f^t(x) = \int_0^x R_{1:t,t}(y)\,dy + \int_x^1 R_{0:t-1,t-1}(y)\,dy.$$

where $R_{st}(y) = \mathbb{E}\,\partial_{h_y^s} \sigma_y^t$.

Note: closed in terms of Σ and f, or alternatively C and R.

<u>Next</u>: Glauber dynamics w. randomized updates.

- Pick u.a.r. spin at each step.
- Alternatively each spin has an exponential clock.

Here we ~~[illegible]~~ fix a spin & describe cts. time process of its spin and local field $\{\sigma^t\}_{t\geq 0} \cup \{h^t\}_{t\geq 0}$, $t \in \mathbb{R}_+$.

- Sample Poi process of update times of rate 1 on $\mathbb{R}_+$
  $P = \{t_1, t_2, \dots\}$.

Then σ^t reacts to h^t at the times in P:

$$\sigma^t = \begin{cases} c(h^{t_i}, u^i) & \text{if } t = t_i \in P \\ \sigma^{t_i} & \text{if } t \in (t_i, t_{i+1}). \end{cases}$$

$$h^t = G^t + \mathbb{E}_{P'}\left[\langle \sigma^{P'}, R(P') \rangle\right].$$

Above: • $\{G^t\}_{t\geq 0}$ is a G.P. of cov. $C(s,t) = \mathbb{E}\,\sigma^s \sigma^t$.

- The expectation is only w.r.t. P', a Poi process of rate 1 on $[0,t]$, indep. of P and of everything else.
- $\sigma^{P'} \equiv \{\sigma^s\}_{s\in P'}$; if $P' = \emptyset$, $\sigma^{P'} \equiv 0$.
- $R(P')$ takes a finite coll. of pts. $P' = \{s_1, \dots, s_k\}$ and outputs a vector in $\mathbb{R}^k$:

$$R(P') = \left\{ \mathbb{E}\left[\partial_{h^{s_i}} \sigma^{s_k} \,\middle|\, P = P'\right] \right\}_{i=1}^{k}$$

<u>Note</u>: also closed in terms of C and R.

<u>Note 2</u>: the integrals for seq. dynamics and $\mathbb{E}_{P'}$ here correspond to averages w.r.t. the update times of the remaining spins.

(2) Rest of talk: pf ideas.

Here we go over carefully a simple example: $T=1$ pass through the spins.

- We start by simplifying the eqns. in this case.

$T=1 \Rightarrow C_{1,1}(y) = \mathbb{E}\,\sigma_y^1 \sigma_y^1 = 1$; similarly $C_{0,0}(y)=1$.

$$\Rightarrow \Sigma^{(1)}(x) = [1] \text{ for all } x \in [0,1].$$

Moreover, $$f'(x) \equiv f(x) = \int_0^x R_{1,1}(y)\,dy + \overbrace{\int_x^1 \underbrace{\cancel{R_{0,0}(y)}}_{\mathbb{E}\,\partial_{h_y^0}\sigma_y^0 = 0}\,dy}^{=0}$$

$$= \int_0^x \mathbb{E}\,\partial_{h_y^1}\sigma_y^1\,dy.$$

$$\Rightarrow h_x^1 = G + \sigma_x^0 \int_0^x \mathbb{E}\,\partial_{h_y^1}\sigma_y^1\,dy, \quad \text{where } G \sim N(0,1).$$

Let's try to derive this. Recall that this ~~is~~ is the effective process; at finite N we have, for $i = xN$,

$$h_i^1 = \sum_{j<i} J_{ij}\sigma_j^1 + \sum_{j \geq i} J_{ij}\sigma_j^0.$$

<u>Observation #1</u>: $\sum_{j\geq i} J_{ij}\sigma_j^0 \sim N(0, 1-x)$, and indep. of $\sum_{j<i} J_{ij}\sigma_j^1$.

So suffices to show that $\sum_{j<i} J_{ij}\sigma_j^1 \sim \sqrt{x}\,G + \sigma_x^0 \int_0^x R_{11}(y)\,dy$

Cavity method idea: all of these spins σ_j^1 are biased in the direction of σ_i. Once we remove this bias it will become Gaussian.

<u>Problem</u>: σ_j^1 is non-differentiable function of σ_i.

<u>Idea</u>: The <u>law</u> of σ_j^1 is still smooth w.r.t. σ_i.

In fact, even if we average only over the randomness of the random seeds, its already smooth (at least at finite temp.).

So we consider the Doob decomposition: let $\overline{\sigma}(h) = \mathbb{E}_{\mathcal{U}}\, \sigma(h, \mathcal{U})$, and $\overline{\sigma}_j' = \overline{\sigma}(h_j')$

$$\underbrace{\sum_{j<i} J_{ij}(\sigma_j' - \overline{\sigma}_j')}_{} + \sum_{j<i} J_{ij}\overline{\sigma}_j'$$

this is a Martingale wrt. the filtration $\mathcal{F}_j = \sigma(\{J\} \cup \{\mathcal{U}_k'\}_{k=1}^{j})$.

By MG CLT, converges to $\cancel{N(0, \ldots)}$ $N(0, \underbrace{\sum_{j<i} J_{ij}^2 (1-(\overline{\sigma}_j')^2)}_{})$

we know converges to $\int_0^x \mathbb{E}(1-(\overline{\sigma}_y')^2)dy$ by prop. of chaos, proved by Gronwall argument.

So suffices to show that $\sum_{j<i} J_{ij}\overline{\sigma}_j' \sim \sqrt{\int_0^x \mathbb{E}(\overline{\sigma}_y')^2 dy}\; G + \overline{\sigma}_y \int_0^x R_1 dy$

But now this is smooth!

- For each j, expand $\overline{\sigma}_j'$ in terms of J_{ij}.

$$\sum_{j<i} J_{ij}\overline{\sigma}_j' = \sum_{j<i} J_{ij}\overline{\sigma}_j'\big|_{J_{ij}=0} + \sum_{j<i} J_{ij}^2 (\partial_{J_{ij}} \overline{\sigma}_j')\big|_{J_{ij}=0} + O\left(\frac{1}{\sqrt{N}}\right).$$

<u>Claim:</u> $\sum_{j<i} J_{ij}\overline{\sigma}_j'\big|_{J_{ij}=0}$ is a M.G. w.r.t. the filtration

$\mathcal{G}_j = \{J_{kl} : i \notin \{k,l\}\} \cup \{J_{i1}, J_{i2}, \ldots, J_{ij}\} \cup \{\mathcal{U}_k'\}_{k=1}^{N}$ ✓

So this indeed becomes Gaussian $N(0, \frac{1}{N}\sum_{j<i} \mathbb{E}(\overline{\sigma}_j')^2) \to N(0, \int_0^x \mathbb{E}(\overline{\sigma}_y')^2 dy)$

So now it suffices to show that

$$\sum_{j<i} J_{ij}^2 \left(\partial_{J_{ij}} \bar{\sigma}_j^1\right)\Big|_{J_{ij}=0} \approx \sigma_i^0 \int_0^x \mathbb{E}\,\partial_{h_y^1} \sigma_y^1 \, dy.$$

But recall that $\bar{\sigma}_j^1 = \bar{c}(h_j^1)$. Hence

$$\partial_{J_{ij}} \bar{\sigma}_j^1 = \left(\partial_{h_j^1} \bar{\sigma}_j^1\right)\left(\partial_{J_{ij}} h_j^1\right).$$

Now $h_j^1 = \sum_{k<j} J_{kj}\sigma_k^1 + \sum_{k\geq j} J_{kj}\sigma_k^0$. But recall $i>j$, so J_{ij} appears in 2^{nd} term. It also doesn't appear in first term due to causality.

$$\Rightarrow \partial_{J_{ij}} h_j^1 = \sigma_i^0.$$

$$\Rightarrow \sum_{j<i} J_{ij}^2 \left(\partial_{J_{ij}} \bar{\sigma}_j^1\right)\Big|_{J_{ij}=0} = \sum_{j<i} \underbrace{J_{ij}^2}_{\approx 1/N} \left(\partial_{h_j^1} \bar{\sigma}_j^1\right) \sigma_i^0$$

$$\to \sigma_i^0 \int_0^x \mathbb{E}\,\partial_{h_y^1} \bar{\sigma}_y^1 \, dy$$

$$= \sigma_i^0 \int_0^x \mathbb{E}\,\partial_{h_y^1} \sigma_y^1 \, dy.$$

Summary: we extract the randomness of the seeds by Martingale CLT. Then we get smooth functions of i. Once we remove the bias in the direction of i, the rest is again Gaussian, and both variances combine to variance 1. The 1^{st} order bias term gives the memory term. → way to make dynamical cavity method rigorous.

For general passes and randomized order: MG → "approx MG", analyzed as Lyndenberg.
Taylor expansion is more complicated: use chain rule & ind. to analyze leading terms.