Sequential Dynamics in Ising Spin Glasses

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Joint work with
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Challenge: characterizing dynamics in random structures

- Given a random Hamiltonian $H(\sigma, \mathcal{R}), \sigma \in \Sigma$, $\mathcal{R}$ encodes randomness of the model.
- Suppose an iterative algorithm $\mathcal{A}$ is implemented sequentially $\sigma^t = \mathcal{A}(\mathcal{R}, \sigma_1, \ldots, \sigma^{t-1}), t = 1, 2, \ldots,$ with fixed (random) start σ^0.
- Examples: Markov Chain Monte Carlo/Glauber dynamics, Greedy, Simulated Annealing, etc.
- **Goal:** Characterise $H(\sigma_t, \mathcal{R})$ for "large" t.
- Verify that for bounded t, say $t = N^{O(1)}$, where N is model dimension.

$$H(\sigma_t, \mathcal{R}) < \max_{\sigma} H(\sigma, \mathcal{R})$$

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Dynamics in spin glasses

- This work: computing the dynamics $H(\sigma_t, \mathcal{R})$ for spin glasses, specifically Sherrington-Kirkpatrick (2-body) N-spin model for linear time scale $T = O(N)$.
- The remainder of this and next talk:
  - Model background
  - Main results
  - Prior literature
  - Algorithmic implications
  - **Part II** by Fran Pernice. Proof approaches: (i) Gaussian conditioning principle (ii) Cavity argument

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Ising spin glasses. Model description

- Given $J = (J_{ij}, 1 \le i < j \le N)$ i.i.d. $\mathcal{N}(0, 1/N)$ – Gaussian symmetric matrix.
- Sherrington-Kirkpatrick model (2-spin model).

$$H(\sigma, J) = \langle \sigma, J\sigma \rangle = \sum_{ij} J_{ij}\sigma_i\sigma_j, \qquad \sigma \in \{\pm 1\}^N.$$

- p-spin model: $J = (J_{i_1,\dots,i_p}) \in \mathbb{R}^{N^{\otimes p}}$ i.i.d. $\mathcal{N}\left(0, \frac{1}{N^{\frac{p-1}{2}}}\right)$.

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Ground states and free energy

Parisi [1980], Talagrand [2006]

- The limit ground state (Parisi constant)

$$\lim_N \frac{1}{N} \max_{\sigma \in \{\pm\}^N} H(\sigma, J) \triangleq \eta_{p,\text{Parisi}}$$

exists and can be computed via PDE.

- The limit free energy

$$\lim_N \frac{1}{N} \log \left(\sum_{\sigma \in \{\pm\}^N} \exp\left(-\beta H(\sigma, J)\right) \right) \triangleq f(p, \beta)$$

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Fields

- Given $\sigma \in \{\pm 1\}^N$ define the field at spin i as

$$h_i = h_i(\sigma) \triangleq \sum_{j \neq i} J_{ij} \sigma_j.$$

- Note:

$$H(\sigma) = \sum_i \sigma_i h_i.$$

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Dynamics. Greedy

Order of updated spins $i(t)$:

(a) i.i.d. uniform random $i(t) \in [N]$.

(b) Scanning dynamics: $i(1) = 1, i(2) = 2, \dots, i(N) = N, i(N+1) = 1, i(N+2) = 2, \dots$ etc. T scans $= TN$ updates.

Greedy.

1. Initialize $\sigma^{(0)} \in \{\pm 1\}^N$.
2. For $t = 1, 2, \dots$ update one coordinate $i(t)$ greedily, keep others the same

$$\sigma^{(t)}_{i(t)} = \arg\max_{\sigma = \pm 1} \sigma h^t_i = \mathrm{Sign}\left(h^t_i\right)$$

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Dynamics. Glauber

Glauber dynamics.

1. Initialize $\sigma^{(0)} \in \{\pm 1\}^N$.
2. For $t = 1, 2, \dots$ update one coordinate $i(t)$ randomly, keep others the same

$$\mathbb{P}\left(\sigma_{i(t)}^{(t)} = \pm 1\right) = \frac{\exp(\pm\beta h_i)}{\exp(\beta h_i) + \exp(-\beta h_i)}$$
$$\sigma_j^{(t)} = \sigma_j^{(t-1)}, \qquad j \neq i(t).$$

General dynamics: any randomized rule of the form $\mathbb{P}(\sigma_i = 1) = c(h_i^t)$ for some fixed function $c(\cdot)$.
$c(x) = (1/2)(\tanh(\beta x) + 1)$ for the Glauber dynamics.

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Main result. Scanning dynamics, informally

- $\sigma^t \in \{\pm 1\}^N$ – spin configuration at the end of scan $t = 1, 2, \ldots, T$.
- Fix $x \in [0,1]$. We give a recursive characterization of the limiting distribution of $\left(\left(\sigma^t_{xN}, h^t_{xN}\right), 1 \le t \le T\right)$ for every constant T

$$\left(\left(\sigma^t_{xN}, h^t_{xN}\right), 1 \le t \le T\right) \stackrel{\text{weakly}}{\longrightarrow} \left(\left(\sigma^t_x, h^t_x\right), 1 \le t \le T\right),$$

for a certain r.v. $\left(\left(\sigma^t_x, h^t_x\right), 1 \le t \le T\right) \in \{\pm 1\}^T \times \mathbb{R}^T$ which we characterize next.
- Asymptotic expected energy achieved at time TN (after T scans) satisfies

$$\lim_N \frac{1}{N} \sum_i \sigma_i^T h_i^T = \sum_{t \le T} \mathbb{E} \int_0^1 (\sigma^t_x - \sigma^{t-1}_x) h^t_x dx.$$

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Main result. Scanning dynamics for T scans, formally

- Fix a psd $K \in \mathbb{R}^{T \times T}$ and a sequence of vectors $v^t \in \mathbb{R}^t, 1 \le t \le T$.
- Let $G = (G^1, \ldots, G^T) \stackrel{d}{=} \mathcal{N}(0, K)$.
- Define $((\sigma^1, h^1), \ldots, (\sigma^T, h^T)) \in \{\pm 1\}^T \times \mathbb{R}^T$ recursively via $\sigma_0 = \pm 1$ u.a.r. and

$$\begin{cases} h^t & = G^t + \langle v^t, (\sigma_0, \ldots, \sigma_{t-1}) \rangle, \\ \sigma^t & = c(h^t, U^t). \end{cases}$$

- For each $K \in \mathbb{R}^{T \times T}, v = (v_1, \ldots, v_T) \in \mathbb{R} \times \mathbb{R}^2 \times \cdots \mathbb{R}^T$ we obtain a stochastic process $(\sigma^t(\omega), h^t(\omega)), 1 \le t \le T$.

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- Define $((\sigma^1, h^1), \dots, (\sigma^T, h^T)) \in \{\pm 1\}^T \times \mathbb{R}^T$ recursively via $\sigma_0 = \pm 1$ u.a.r. and

$$\begin{cases} h^t & = G^t + \langle v^t, (\sigma_0, \dots, \sigma_{t-1})\rangle, \\ \sigma^t & = c(h^t, U^t). \end{cases}$$

- For each $K \in \mathbb{R}^{T\times T}, v = (v_1, \dots, v_T) \in \mathbb{R} \times \mathbb{R}^2 \times \cdots \mathbb{R}^T$ we obtain a stochastic process $(\sigma^t(\omega), h^t(\omega)), 1 \le t \le T$.

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$$\mathcal{C}(K,v) = \mathbb{E}\sigma^s\sigma^t, \qquad \text{Correlation function, } 0 \le s,t \le T,$$
$$\mathcal{R}(K,v) = \mathbb{E}G^s\sigma^t, \qquad \text{Response function, } 0 \le s,t \le T.$$

- Consider an ODE uniquely defining $K(x) \in \mathbb{R}^{T\times T}, v(x) \in \mathbb{R}\times\mathbb{R}^2\times\cdots\mathbb{R}^T, x \in [0,1]$:

$$\frac{dK(x)}{dx} = \mathcal{C}_{1:T,1:T}(K(x),v(x)) - \mathcal{C}_{0:T-1,1:T-1}(K(x),v(x)),$$
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Boundary condition

$$K(0) = \int_0^1 \mathcal{C}_{0:T-1,0:T-1}(K(x), v(x))dx,$$
$$v(0) = \int_0^1 \mathcal{R}_{0:T-1,T-1}(K(x), v(x))dx.$$

Main result. Scanning dynamics for T scans, formally

Theorem

Fix $x_1, \ldots, x_R \in [0,1]$. Then the distribution of

$$(\sigma^t_{x_r N}, h^t_{x_r N}, 1 \le t \le T, 1 \le r \le R) \overset{\text{weakly}}{\longrightarrow} (\sigma^t_{x_r}, h^t_{x_r}, 1 \le t \le T).$$

converges to the independent product of $(\sigma^t_{x_r}, h^t_{x_r}, 1 \le t \le T), 1 \le r \le R$.

Main result. Scanning dynamics for T scans, formally

Corollary

Let $(\sigma_x^t, h_x^t, 1 \leq t \leq T) \in \{\pm 1\}^T \times \mathbb{R}^T$ *be distributed according the stochastic process driven by* $(K(x), v(x))$ *solving this ODE. For every "nice" update function c and (pseudo-Lipschitz) test function* $\phi : \{\pm 1\}^T \times \mathbb{R}^T \to \mathbb{R}$

$$\lim_N \frac{1}{N} \sum_{1 \leq i \leq N} \phi(\sigma_i^t, h_i^t, 1 \leq t \leq T) = \int_0^1 \mathbb{E}\phi(\sigma_x^t, h_x^t, 1 \leq t \leq T) dx.$$

Dynamics from fixed (warm) starts. Prior art

- Physics derivation of equations for Langevin dynamics for spherical spin glasses. Discovery of ”aging” Cugliandolo and Kurchan [93]
- Rigorous verification of CK equations Ben Arous, Dembo and Guionnet, [95],[97],[01]

$$C(s,t) \sim \frac{1}{\left(\frac{t}{s}\right)^{\frac{3}{4}}}, \qquad 0 << s << t.$$

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Overlap Gap Property (OGP)

Theorem (Chen, G, Panchenko, Rahman [19], G, Jagannath, Kizildag [23])

p-spin model exhibits OGP for $p = 4, 6, 8, \geq 10$. *Specifically, there exists* $\eta_{p,\mathrm{OGP}} < \eta_{p,\mathrm{Parisi}}$ *s.t. for every* $\sigma_1, \sigma_2 \in \{\pm\}^N$ *satisfying* $H(\sigma_1), H(\sigma_2) \geq \eta_{p,\mathrm{OGP}}$ *it holds*

$$\frac{1}{N}\langle \sigma_1, \sigma_2 \rangle \notin (\nu_1, \nu_2)$$

for some $0 < \nu_1 < \nu_2 < 1$.
OGP is an obstruction to stable algs.

OGP is a barrier to sequential dynamics at linear time scale

- Consider smoothed dynamics where $\tau_i \in [-1,1]$ using smooth approximation ϕ of the $\mathrm{sign}(x)$ function. Consider the implied dynamics $\tau^t \in [-1,1]^N, h^t \in \mathbb{R}^N$.

Theorem (This work)

(a) *Suppose $J \approx \hat{J}$. Then for every $T = O(1)$ (linear time scale), $\tau^T(J) \approx \tau^T(\hat{J})$. Namely τ^T is stable as a function of disorder J. As a result $H(\tau^T) \leq \eta_{p,\mathrm{OGP}}$ whp.*

(b) *Suppose $\phi \approx \mathrm{sign}(x)$. Then $(\sigma^T, h^T) \overset{d}{\approx} (\tau^T, h^T)$. Then $H(\sigma^T) \approx H(\tau^T) \leq \eta_{p,\mathrm{OGP}}$ - OGP is a barrier to sequential dynamics at linear time scale.*

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